Homework 4: Duality and Linear Programming

Instructions: Show all applied concepts and steps to support final answers.

Problem 1: *(10 points)* Convert the following linear programming problem to standard form:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 0 \leq x_1 \leq 2 \\ & x_1 + x_2 \leq 3 \\ & x_1 + 2x_2 \leq 5 \\ & x_2 \geq 0 \end{aligned}$$

Problem 2: *(10 points)* Consider the optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i}^{n} c_i \, |x_i| \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

where $c_i \neq 0$. Convert this problem into an equivalent standard form linear programming problem.

Hint: Given any $x \in \mathbb{R}$ we can find unique numbers x^+, $x^- \in \mathbb{R}$, $x^+, x^- \geq 0$, such that $|x| = x^+ + x^-$ and $x = x^+ - x^-$.

Problem 3: *(10 points)* Suppose that a wireless broadcast system has n transmitters. Transmitter j broadcasts at a power of $p_j \geq 0$. There are m locations where the broadcast is to be received. The path gain from transmitter j to location i is $g_{i,j}$, that is, the power of the signal transmitted from transmitter j received at location i is $g_{i,j}p_j$. The total power received at location i is the sum of the powers received from all the transmitters. Formulate the problem of finding the minimum sum of the powers transmitted subject to the requirement that the power received at each location is at least P.

Problem 4: *(30 points)* Derive the KKT conditions for the standard form linear program and its dual, and show that the two sets of KKT conditions are identical.

Primal	**Dual**
$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$	$$\begin{aligned} \max_{\mathbf{y}} \quad & \mathbf{y}^\top \mathbf{b} \\ \text{subject to} \quad & \mathbf{y}^\top \mathbf{A} \leq \mathbf{c}^\top \end{aligned}$$

Hint: Take $\mathbf{y}$ as the Lagrange multipliers for the equality constraints in the primal, and $\mathbf{x}$ as the Lagrange multipliers for the inequality constraints in the dual.

Problem 5: *(20 points)* Consider the optimization problem:

$$\begin{aligned} \min_{x,y} \quad & -xy \\ \text{subject to} \quad & (x-3)^2 + y^2 = 5 \end{aligned}$$

a) Use the first order necessary conditions and second order sufficient conditions to identify the global minimum $\mathbf{x}^\star = (x^\star, y^\star)$ of this problem, and the associated value of the Lagrange multiplier.

b) Derive the dual function for this optimization problem.
Hint: For equality constrained problems, will $q(\boldsymbol{\lambda})$ *depend on only* $\boldsymbol{\lambda}$*, only* $\mathbf{x}^\star$*, or both* $\boldsymbol{\lambda}$ *and* $\mathbf{x}^\star$*?*

Problem 6: *(20 points)*

a) For what range of values of λ does the following unconstrained optimization problem have a solution?:

$$\min_{x,y} \quad -xy - \lambda\big((x-3)^2 + y^2 - 5\big).$$

b) Derive the optimal value of the unconstrained optimization problem in part a) as a function of λ.

c) Find the value of λ at which this optimal value is maximized.