ME/AER 676 Robot Modeling & Control
Spring 2023

Rotations

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- If $\det T_B^A = 1$, then the ordering of the basis of B satisfies some order defined by basis of B

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- Since $\left(T_B^A\right)^T T_B^A = I$, the magnitude of vectors doesn't change, only the direction
- Therefore, these transformations are rotations, and they form the special orthogonal group SO(3) (in 3D).

SO(3)

Definition (Special Orthogonal group in 3D)

The Special Orthogonal Group SO(3) is the set of matrices $R \in \mathbb{R}^{3\times 3}$ such that

$$R^T R = Id, \text{and } \det R = 1$$

.

$SO(3)$ is known as the orientation group **and** the rotation group.

Exercise: Show that SO(3) forms a group under matrix multiplication.

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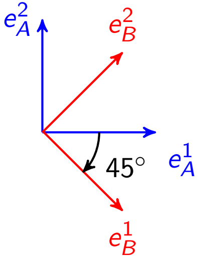
$$e_B^1 = \frac{1}{\sqrt{2}} e_A^1 - \frac{1}{\sqrt{2}} e_A^2$$

$$e_B^2 = \frac{1}{\sqrt{2}} e_A^1 + \frac{1}{\sqrt{2}} e_A^2$$

$$e_B^3 = 1 \cdot e_A^3$$

$$\implies R = T_B^A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Constructing Coordinate Frames

- Given any three non-collinear 3D vectors, we may define a rotation matrix by Gram-Schmidt orthonormalization.
- Therefore, four non-coplanar points a, b, c, d on a rigid body are enough to define a cartesian frame fixed to the body
  - One point becomes the origin
  - The remaining three points define a vector relative to the origin point
  - orthonormalize vectors to get vectors defining cartesian frame and its orientation
  - origin + rotation matrix = coordinate of body (frame)

SO(3)

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- The G-Torsor nature is why SO(3) is called both the rotation group and the orientation group.
- Assigning coordinates to an orientation is the same as defining the rotation that generates that frame relative to a reference.

Basic Rotations

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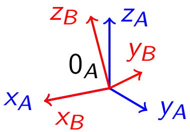

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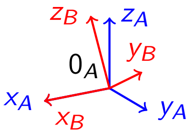

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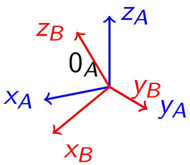

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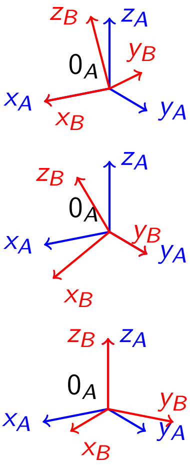
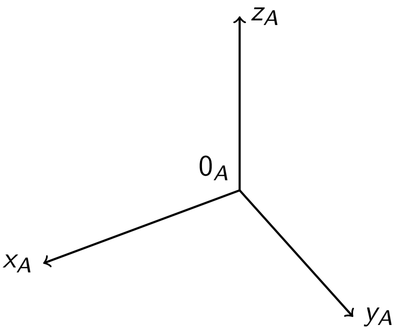


General Rotations

- We can construct a general rotation using a sequence of basic rotations. (Compare to Euclidean space)
- So, orientation coordinates can be derived by sequences of basic rotations (combined through multiplications).
- For Euclidean vector spaces, the order of a sequence of (vector space) operations didn't matter: $v + w = w + v$.
- For rotations, they do. In general, $R_1 R_2 \neq R_2 R_1$.
- One interpretation of the two orders of multiplication is extrinsic vs. intrinsic rotations (next slide)

Extrinsic vs Intrinsic Rotations



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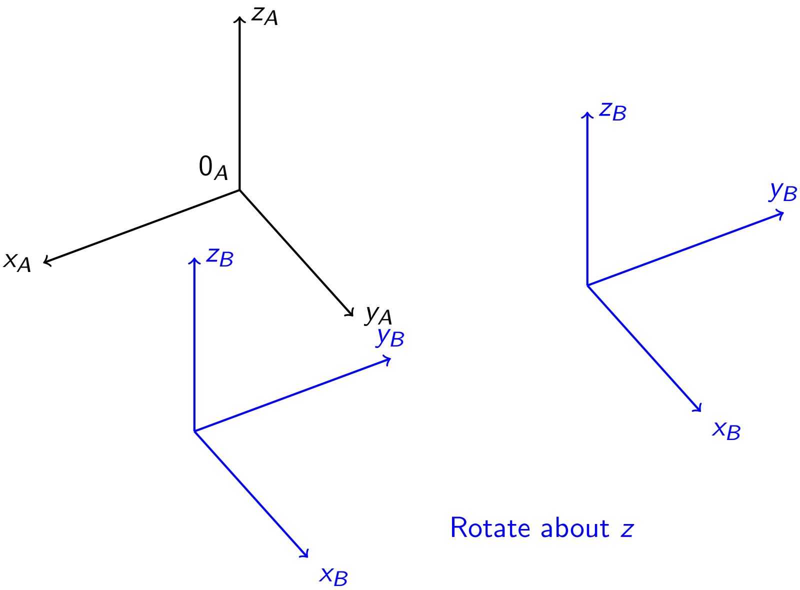
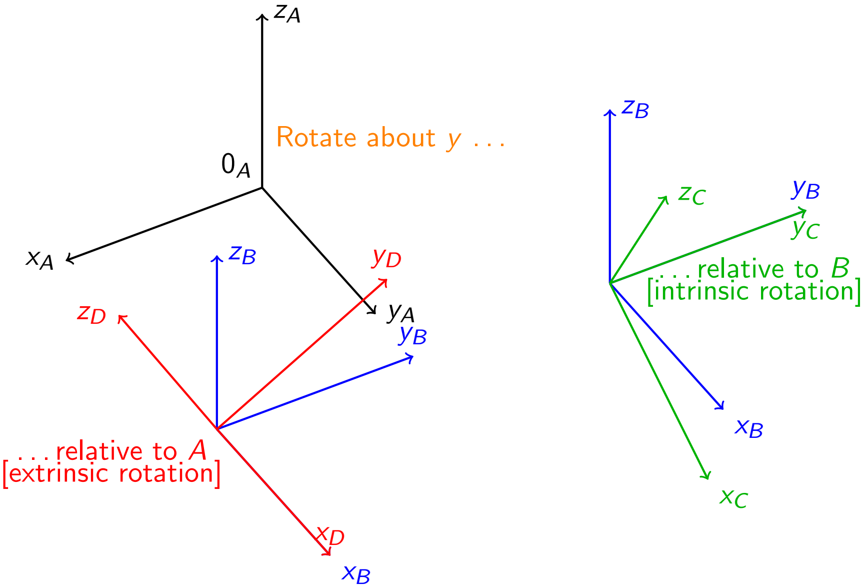


Rotate about z

Extrinsic vs Intrinsic Rotations



Extrinsic vs Intrinsic Rotations

- A first rigid motion corresponding to rotation R_1 relative to a frame A produces frame B
- A second rigid motion rotation R_2 can be applied relative to either A or B.
- When applied relative to B, the second rotation is an intrinsic rotation. $R = R_1 R_2$.
- When applied relative to A, the second rotation is an extrinsic rotation. $R = R_2 R_1$.

Euler Angles

- Euler angles use three basic rotations to define any orientation
- Many possible conventions based on
  - Choice of axes of three basic rotations
  - Sequence of extrinsic vs intrinsic rotations
- See notes and texts for more details

Axis-Angle Formula

Alternatively, we may represent a rotation as a single angle of rotation θ and an axis $\mathbf{k} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T$, leading to a formula for R:

$$R = I + (\sin\theta)K + (1 - \cos\theta)K^2 \tag{1}$$

where

$$K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix},$$

and $\mathbf{k}$ has unit norm.

The notes provide another formula where we represent the vector $\mathbf{k}$ using two angles α and β that define basic rotations to produce R .

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Which rotation R_B^C below correctly defines the new orientation of B relative to orientation C?

1. $R_B^C = R_B^A R_A^C$
2. $R_B^C = R_B^A R_C^A$
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How would you pick the right transformation? Why did we not consider R_A^B?

Change-of-Basis For Orientations

- For example, imagine you, a driver, and a passenger in a car. Your orientation frames are aligned: Forward: x, upwards: z.
- When the car stops, the passenger opens the door spins to their right ($R_C^A = R_{z,-90^\circ}$)
- You lean back in your driver's seat ($R_B^A = R_{y,-20^\circ}$)
- What is your orientation according to the passenger?
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To change the coordinates of vectors from A to C, we must pre-multiply by $\left(R_C^A\right)^{-1} = R_A^C$. So,

$$R_B^C = R_A^C R_B^A$$

.

Change-of-Basis For Orientations

Alternatively, The rigid motion in A corresponding to moving to frame B is R_B^A; the rigid motion in frame C corresponding to moving to frame A is R_A^C.

The combined rigid motion in C is achieved by first moving by R_B^A **in C**, then moving the result by R_A^C.
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Instead of orientation R_B^A in frame A, what if we define **rotation** R^A in frame A.
How do we represent this rotation in frame C?

Change-of-Basis For Rotations

- The rotation R^A is relative to frame A.
- Any generic orientation P has coordinates R_P^A in frame A
- Rotating this orientation results in a new orientation $R^A R_P^A$ in frame A:

$$R_P^A \mapsto R^A R_P^A$$

- But, note that $R_P^A = R_C^A R_P^C$
- Therefore :

$$R_C^A R_P^C \mapsto R^A R_C^A R_P^C, \text{or}$$

$$R_P^C \mapsto \left(R_C^A\right)^{-1} R^A R_C^A R_P^C, \text{or}$$

- Therefore, a rotation R^A in frame A becomes a rotation

$$R^C = \left(R_C^A\right)^{-1} R^A R_C^A$$

in frame C.

Summary

- Rotations of bodies (equivalently, cartesian frames) correspond to a specific linear transformation
- The matrix representing any rotation belongs to SO(3), a group under matrix multiplication
- A rotation defines an orientation (part of the coordinates of a frame), given a reference orientation.
- We may use basic rotations defined about axes to construct any orientation
- Changing reference frames requires changing orientations, and also rotations, appropriately