ME/AER 676 Robot Modeling & Control
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Velocities of Frames

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  $d \in \mathbb{R}^3$, $R \in \mathrm{SO}(3)$

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- However, the coordinate (d, R) is not a vector!

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- However, if x belonged to a group, we can't define a derivative this way

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- The 'velocity' would require us to take the limit as $h \to 0$ of the ratio of $\Delta R(h)$ and some measure of the size of $\Delta R(h)$.

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- Physically, the vector $\omega = \begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}^T$ defines the instantaneous angular velocity in frame $\{0\}$

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- So, when a task is $x(t) = (d(t), R(t)) \in \mathbb{R}^3 \times SO(3)$, its velocity i s

$$\xi \in \mathbb{R}^6 = \underbrace{\mathbb{R}^3}_{\text{linear velocity}} \times \underbrace{\mathbb{R}^3}_{\text{angular velocity}}$$

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  - As the three derivatives of the three numbers used to parametrize SO(3).

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- Columns of $J(q)$ of geometric Jacobian are derived geometrically