ME/AER 676 Robot Modeling & Control
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Twists and Wrenches

Hasan A. Poonawala

Department of Mechanical Engineering
University of Kentucky

Email: hasan.poonawala@uky.edu
Web: https://www.engr.uky.edu/~hap

Twist $\mathcal{V}$ vs ξ

- We've seen that we can represent the velocity of a frame $\{n\}$ using

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix},$$

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- We can, and that's what body twists are (with a twist)

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- $$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

S

Angular Velocity Frames

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- The equations may therefore be rewritten as

  $$\dot{R}(t) = R(t)[\omega_b]$$

Angular Velocity Frames

Twist

- We've seen:

$$\omega \in \mathbb{R}^3 \to [\omega] \in \mathfrak{so}(3) \to \dot{R}(t) \to R \in \mathrm{SO}(3)$$

- These transformations are well-defined because SO(3) is a *Lie* group: a group with a manifold structure
- Any Lie group has a similar set of manipulations
- SE(3) (homogenous transformations) are also a Lie group
- The 'angular velocity' corresponding to SE(3) is a *twist*
- Twists for *SE* are not as intuitive as angular velocities for SO(3).

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- If $\omega \in \mathbb{R}^3$ represent velocities for SO(3), twists $\mathcal{V}$ represent velocities for SE(3)
- Consider a homogenous tranformation $T(t) \in SE(3)$ representing a rigid body pose of $\{b\}$ in $\{s\}$:

$$T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \quad (T = \underbrace{T_b^s}_{\text{RMC}} = \underbrace{T_{sb}}_{\text{MR}} = \underbrace{H_b^s}_{\text{HP}}) \tag{2}$$

Body Twist

- If the angular velocity in the body frame is ω_b, and the velocity of the origin is v_b, then

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- The body twist has simple physical meaning: instantaneous angular velocity of $\{b\}$ as seen in $\{b\}$, and instantaneous velocity of origin of $\{b\}$ as seen in $\{b\}$

Spatial Twist

- We can convert the body twist $\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$ into a spatial twist $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}$

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- However, v_s is **not** the velocity of the origin of $\{b\}$ as viewed in $\{s\}$

- v_s a fictitious velocity of the origin of $\{s\}$ as if the space frame $\{s\}$ was rotating about axis ω_s that passes through origin of $\{b\}$ (Fig 3.17 in MR).

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- However, only for the body twist do we see a clear physical interpretation of its component ω and v in terms of frame velocities

All The Velocities And Jacobians

If we represent frame velocity using body twist $\mathcal{V}_b$ or spatial twist $\mathcal{V}_s$, we need a body Jacobian J_b and spatial Jacobian J_s to implement velocity kinematics.

Name	Task velocity	Spatial Twist	Body Twist
angular velocity	ω_0	ω_0	ω_3
linear velocity ...	v_0	$v_0 - \omega_0 \times o_{30}$	v_3
... of point	o_3	o_0	o_3
Jacobian	Task	Spatial	Body
Symbol	$J(q)$	J_s	J_b