ME 599/699 Robot Modeling & Control
Fall 2021

Optimal Control

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Optimal Control

In continuous time, we have

$$\begin{aligned} \min \quad & J(q(t), u(t)) \\ \text{subject to} \quad & q(t) \text{ satisfies dynamics and state constraints} \\ & u(t) \text{ satisfies input constraints} \end{aligned}$$

We may also formulate discrete time versions of this problem.

(Generalized) Linear Quadratic Regulator

For optimal control problems where

- time is discrete,
- the dynamics are linear, and
- the cost function is quadratic in state and control,

the optimal control problem may be solved in a straightforward way.

These slides are inspired by Sergey Levine's slides.

(Generalized) Linear Quadratic Regulator

At each time $t \in \{0, 1, 2, \ldots, T\}$, we have

$$\mathbf{x}_{t+1} = A_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + a_t; \quad c_t(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

Consider a finite time horizon $t \in \{0, 1, 2, \ldots, T\}$.
Let

$$J = \sum_{t=0}^{T} c_t(x_t, u_t)$$

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The cost for the first $T-1$ time steps are some value that is effectively constant at time T, so that the total cost will be $\mathbf{Q}_T(\mathbf{x}_T, \mathbf{u}_T)$

$$\mathbf{Q}_T(\mathbf{x}_T, \mathbf{u}_T) = \text{const} + \frac{1}{2}\begin{bmatrix}\mathbf{x}_T \\ \mathbf{u}_T\end{bmatrix}^T \mathbf{C}_T \begin{bmatrix}\mathbf{x}_T \\ \mathbf{u}_T\end{bmatrix} + \begin{bmatrix}\mathbf{x}_T \\ \mathbf{u}_T\end{bmatrix}^T \mathbf{c}_T$$

Optimize at T

To find the best $\mathbf{u}_T$, we minimize that expression.

It's gradient w.r.t. $\mathbf{u}_T$ is

$$\nabla_{\mathbf{u}_T} \mathbf{Q}_T(x_T, u_T) = x_T^T \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} + u_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T} + \mathbf{c}_{\mathbf{u}_T}^T, \text{ where}$$

$$\mathbf{C}_T = \begin{bmatrix} \mathbf{C}_{\mathbf{x}_T,\mathbf{x}_T} & \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} \\ \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} & \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T} \end{bmatrix}, \quad \mathbf{c}_T = \begin{bmatrix} \mathbf{c}_{\mathbf{x}_T} \\ \mathbf{c}_{\mathbf{u}_T} \end{bmatrix}.$$

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Setting $\nabla_{u_T} Q_T(x_T, u_T) = 0$ we obtain

$$\mathbf{u}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \left(\mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{x}_T + \mathbf{c}_{\mathbf{u}_T} \right) = \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T,$$

which is a linear (well, affine) feedback control.

Cutting to the Chase

- To cut a long story short,

$$\mathbf{Q}_T(\mathbf{x}_T, \mathbf{u}_T) = \mathbf{Q}_T(\mathbf{x}_T, \mathbf{K}_T\mathbf{x}_T + \mathbf{k}_T) = V(\mathbf{x}_T) = \mathbf{x}_T^T\mathbf{V}_T\mathbf{x}_T + \mathbf{x}_T^T\mathbf{v}_T,$$

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  for some appropriate matrix $\mathbf{V}_T$ and $\mathbf{v}_T$ that depends on the problem's parameters.
- Because the dynamics are linear, and costs are quadratic, the same thing repeats at $t = T - 1$

$$\begin{aligned}
\mathbf{Q}_{T-1}(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) &= \mathrm{const} + c_{T-1}(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) + V(\mathbf{x}_T) \\
&= \mathrm{const} + c_{T-1}(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) \\
&\qquad + V\left(A_{T-1}\begin{bmatrix}\mathbf{x}_{T-1} \\ \mathbf{u}_{T-1}\end{bmatrix} + a_{T-1}\right) \\
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- This nice structure persists till $t = 0$

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3. The function $Q_t(\mathbf{x}_t, \mathbf{u}_t)$ is known as the Q-function in reinforcement learning
4. V is the value function (we minimize, RL maximizes)

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- Instead, some approaches compute $V_T / V(t)$ directly (Hamilton-Jacobi-Bellman equations)
- These methods require knowing dynamics and reward functions

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- Main challenge is in trading-off learning and optimizing (exploration-exploitation trade-off)

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  - Model-free: Maintain policy π and value V using data

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Terms you will come across

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- Policy gradients (PPO, SAC)
- Optimization (TRPO, iLQR)

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- Most successful approaches use low-level position-based control (impedance or otherwise) on position-based tasks

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- Most papers are opaque about how much human engineering is involved
- My opinion: use to choose controller, not to design control
- My lab: learn NN models from data, design correct controllers for such models