ME 599/699 Robot Modeling & Control
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State Uncertainty as Probability Distributions

Hasan A. Poonawala

Department of Mechanical Engineering
University of Kentucky

Email: hasan.poonawala@uky.edu
Web: https://www.engr.uky.edu/~hap

Uncertain State

Instead of saying our state x has a specific value, say

$$x = \begin{bmatrix} 1 \\ 3.4 \end{bmatrix},$$

we say that the state is a random variable $\mathbf{X}$ with continuous/discrete probability distribution $p_{\mathbf{X}}(x)$.

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For example, let $\mathbf{X}$ be a random variable that can have integer values.
Then, we can speak of the probability that $\mathbf{X} = 4$, or $\mathbf{X} = 19283$, denoted as $p_{\mathbf{X}}(4)$ and $p_{\mathbf{X}}(19283)$ respectively.

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Similarly, a dice D has six outcomes $\{1, 2, 3, 4, 5, 6\}$, and we use five numbers to describe the uncertainty in single rolls of a dice.

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Consider the real interval (a, b)
Then,

$$\Pr(X \in (a, b)) = \int_a^b p_{\mathbf{X}}(x) dX$$

Gaussian Random Variable

A common probability density function is the Gaussian distribution:

$$p_{\mathbf{X}}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

- μ is the average value of the distribution
- σ is the standard deviation.
- σ^2 is called the variance.
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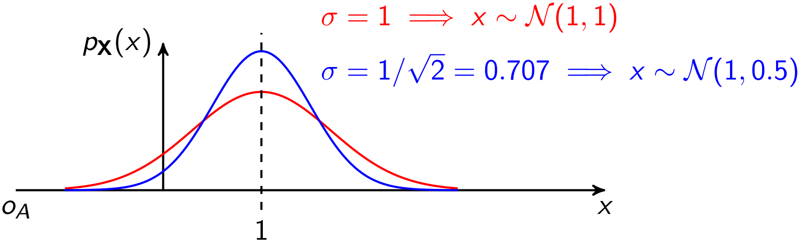
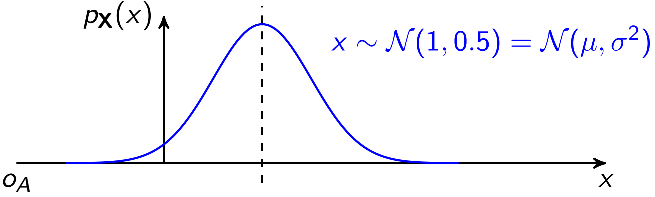
We can plot this function $p_{\mathbf{X}}(x)$:



Uncertainty regions



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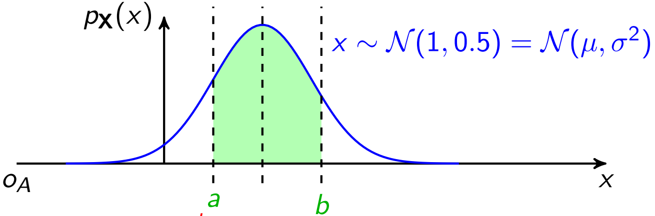
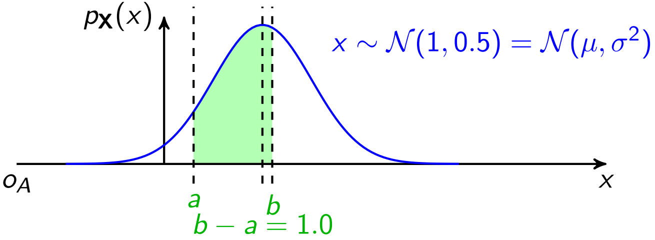


$\Pr(x \in (a, b)) = \int_a^b p_{\mathbf{x}}(x)dx =$ area under curve

Uncertainty regions



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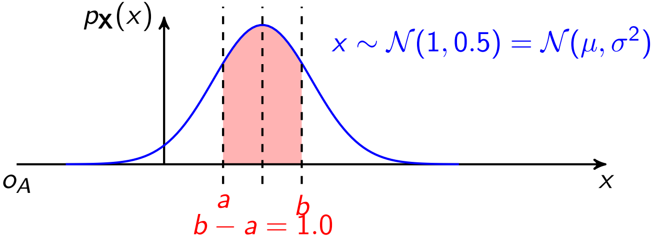
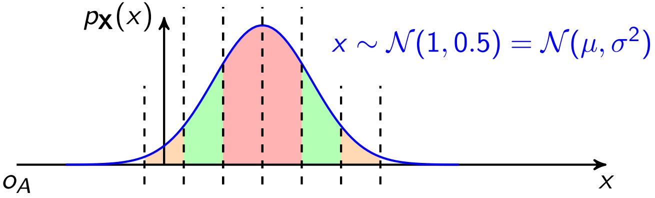


Same interval, larger probability when centered at μ

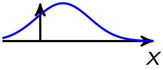
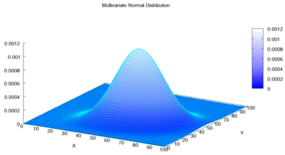
Uncertainty Regions



Consider the interval $I(k) = (\mu - k\sigma, \mu + k\sigma)$:
length $2k\sigma$ centered at μ.

- $\Pr(x \in I(1)) = 0.682$ (red)
- $\Pr(x \in I(2)) = 0.954$ (red + green)
- $\Pr(x \in I(3)) = 0.997$ (red + green + orange)

Multivariate Gaussians

	$\mathbb{R}$-valued Gaussian	$\mathbb{R}^n$-valued Gaussian
$p_{\mathbf{X}}(x)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\frac{1}{(2\pi \det(\Sigma))^{\frac{n}{2}}} \exp^{-\frac{1}{2}\left((x-\mu)^T \Sigma^{-1} (x-\mu)\right)}$
		
$I(k)$	$\left(\frac{x-\mu}{\sigma}\right)^2 \leq k^2$ (interval in $\mathbb{R}$)	$(x-\mu)^T \Sigma^{-1} (x-\mu) \leq k^2$ (ellipse in $\mathbb{R}^n$)

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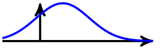
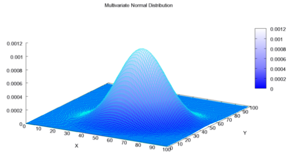
$p_{\mathbf{X}}(x)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$  	$\frac{1}{(2\pi \det(\Sigma))^{\frac{n}{2}}} \exp^{-\frac{1}{2}\left((x-\mu)^T\Sigma^{-1}(x-\mu)\right)}$ 
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Main takeaway: We represent $p_{\mathbf{X}}(x) \sim \mathcal{N}(\mu, \Sigma)$ as an ellipse in X, instead of $p_{\mathbf{X}}(x)$ vs x.

Center of ellipse $\leftrightarrow$ mean μ

Shape/Size of ellipse $\leftrightarrow$ Covariance Σ