ME 599/699 Robot Modeling & Control
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Recursive State Estimation

Hasan A. Poonawala

Department of Mechanical Engineering
University of Kentucky

Email: hasan.poonawala@uky.edu
Web: https://www.engr.uky.edu/~hap

Overview

Overview

Bayesian Methods

Kalman Filter

Other Filters

Bayesian Estimation and Kalman Filtering: Grad Version

State Estimation

Challenge

How do we extract $x(t)$ from our measurements?

There are a few options:

1. Bayesian Inference [Eg. Kalman Filter]
2. Implicitly invert the forward map $y(t) = h(x(t))$ using dynamics and history [Eg. Luenberger Observer]
3. Explicitly invert the forward map to get $x(t) = h^{-1}(y(t))$ [Eg. Encoder-Decoder archs. in ML, Computer Vision]

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We represent this lack of certain knowledge by treating x_t at each time t as a **random variable**.

Uncertain State

Instead of saying our state x has a specific value, say

$$x = \begin{bmatrix} 1 \\ 3.4 \end{bmatrix},$$

we say that the state is a random variable $\mathbf{X}$ with continuous/discrete probability distribution $p_{\mathbf{X}}(x)$.

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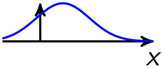
For example, let $\mathbf{X}$ be a random variable that can have integer values.
Then, we can speak of the probability that $\mathbf{X} = 4$, or $\mathbf{X} = 19283$, denoted as $p_{\mathbf{X}}(4)$ and $p_{\mathbf{X}}(19283)$ respectively.

Multivariate Gaussians

	$\mathbb{R}$-valued Gaussian	$\mathbb{R}^n$-valued Gaussian
$p_{\mathbf{X}}(x)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ 	$\frac{1}{(2\pi \det(\Sigma))^{\frac{n}{2}}} \exp^{-\frac{1}{2}\left((x-\mu)^T\Sigma^{-1}(x-\mu)\right)}$ images/Multivariate_Gaussian.p
$I(k)$	$\left(\frac{x-\mu}{\sigma}\right)^2 \leq k^2$ (interval in $\mathbb{R}$)	$(x-\mu)^T\Sigma^{-1}(x-\mu) \leq k^2$ (ellipse in $\mathbb{R}^n$)

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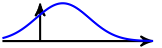
$p_{\mathbf{X}}(x)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ 	$\frac{1}{(2\pi \det(\Sigma))^{\frac{n}{2}}} \exp^{-\frac{1}{2}\left((x-\mu)^T\Sigma^{-1}(x-\mu)\right)}$ images/Multivariate_Gaussian.p
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Main takeaway: We represent $p_{\mathbf{X}}(x) \sim \mathcal{N}(\mu, \Sigma)$ as an ellipse in X, instead of $p_{\mathbf{X}}(x)$ vs x.

Center of ellipse $\leftrightarrow$ mean μ

Shape/Size of ellipse $\leftrightarrow$ Covariance Σ

Bayesian Methods

images/cartoon_state_est.png

Initial uniform belief on x

Observe a door. We model probability of observing door as $p(z|x)$

Update $p(x)$; matches $p(z|x)$

Update $p(x)$ due to motion, peaks ‘flatten’

Observe a door again.

Bayes update makes being at second door highly likely.

Motion update flattens peaks

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Bayes' Rule:

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The updated uncertainty is p(state 'given' measurement), or $p(x|y)$, or p(state 'conditioned on' measurement)

Motion Model

Secondary idea: When the robot moves, $p(x_t)$ and $p(x_{t+1})$ should depend on this motion, even if we never measure anything.

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Given an input u_t at time t, we need a **motion model** $p(x_{t+1}|x_t, u_t)$

In summary, we have two types of updates to our model of the uncertain state:

1. Update due to motion
2. Update due to measurement

Overview

Kalman Filter

The Kalman Filter was developed for discrete-time linear time-invariant dynamical systems driven by noise:

$$x_{t+1} = Ax_t + Bu_t + v_t; \quad y_t = Cx_t + w_t,$$

where v_t and w_t represent zero-mean gaussian noise with covariances V and W respectively.

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At time t, $p(x_t|\mathbf{y}_{0:t}, \mathbf{u}_{0:t-1})$ will also be Gaussian, and our job is to come up with an appropriate mean μ_t and variance Σ_t using inputs and measurements.

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The Kalman filter equations tell us the best way to combine the uncertain motion and uncertain measurement, given these models

Kalman Filter Algorithm

System :

$$x_{t+1} = Ax_t + Bu_t + v_t \tag{1}$$

$$y_t = Cx_t + w_t \tag{2}$$

$$\hat{x}_t \sim \mathcal{N}(\mu_t, \Sigma_t), w_t \sim \mathcal{N}(0, R_w), v_t \sim \mathcal{N}(0, R_v) \tag{3}$$

1. Motion update

$$\mu_t^{pred} = A\mu_t + Bu_t \tag{4}$$

$$\Sigma_t^{pred} = A\Sigma_t A^T + R_v \tag{5}$$

2. Measurement update

$$K_t = \Sigma_t^{pred} C^T \left(C\Sigma_t^{pred} C^T + R_w \right)^{-1} \tag{6}$$

$$\mu_{t+1} = \mu_t^{pred} + K(y_t - C\mu_t^{pred}) \tag{7}$$

$$\Sigma_{t+1} = (I - KC)\Sigma_t^{pred} \tag{8}$$

Example: 1D Point Mass KF

System $m\ddot{q}(t) = u$ has discrete-time state

$$x_t = \begin{bmatrix} q_t \\ v_t \end{bmatrix}$$

where v_t is the velocity of the mass.

The discrete-time dynamics, with discretization time T is

$$x_{t+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ \frac{T}{m} \end{bmatrix} u_t + v_t \tag{9}$$

$$y_t = C x_t + w_t \tag{10}$$

If we measure position, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.
If we measure velocity, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.
If we measure both, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example: 1D Point Mass KF

Performance (Run `Julia` code on Canvas):

- Measuring position using very noisy sensors won't work
- Measuring position using decent sensors works well enough
- Measuring velocity with noisy sensors is terrible
- Measuring velocity with decent sensors is better, but still has the issue of (slow or fast) drift in position over time. This drift is undetected by the filter.
- Conclusion: we need some form of measurement of position

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For the case where the posterior distribution is difficult to describe using a parametric distribution, particle filters are a widely used alternative.

The distribution is represented by a set of samples independently drawn from the distribution.

Particle Filter: Algorithm

1. Move your samples according to the motion model.
2. Weight samples based on how likely they explain the measurement
3. Use the weights to resample from the existing set of points.
4. Return the resampled points as the new set of samples.

Luenberger Observer

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Obstacle

How do we choose L? How do we initialize our estimate?

Luenberger Observer: Analysis

$$\text{L. Obs, :} \quad \frac{d}{dt}\hat{x} = f(\hat{x}, u) + L(y - h(\hat{x}))$$

Good estimation (according to control theorists) occurs when

$$\hat{x}(t) \to x(t) \text{ as } t \to \infty.$$

This property is the same as saying that the origin of system

$$\frac{d}{dt}e(t) = \frac{d}{dt}x(t) - \frac{d}{dt}\hat{h}(t)$$

be asymptotically stable, where $e(t) = x(t) - \hat{x}(t)$.

Luenberger Observer: Analysis

$$\begin{aligned}\frac{d}{dt}e(t) &= \frac{d}{dt}x(t) - \frac{d}{dt}\hat{h}(t) \\ &= f(x,u) - \hat{f}(x,u) - L(y - h(\hat{x})) \\ &= f(x,u) - \hat{f}(x,u) - L(h(x) - h(\hat{x}))\end{aligned}$$

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However, if our system is linear, we get

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&= Ax - A\hat{x} + LC(x - \hat{x}) \\
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If our system is nonlinear, we may be able to linearize to design L and use in an observer, provided $e(t)$ is small enough:

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u) + L(y - h(\hat{x})).$$

Luenberger Observer: Issues

1. Q: What control u do we use?
   - For asymptotic stability, $u = -K\hat{x}$ works when $(A - BK)$ is Hurwitz
   - If we are optimizing a quadratic objective, we get the Linear Quadratic Gaussian problem, with solution $u = -K(t)\hat{x}(t)$.
   - Both cases have a separation principle: can design K and L separately.
2. For nonlinear systems handled by linearization, we need $e(t)$ to be small enough for the L to work. Determining how small is small enough is a separate issue.

Overview

Bayesian Methods

What do we know at time t?

1. A prior distribution $p(x_0)$
2. A history of measurements, $\mathbf{y}_{0:t}$
3. A history of inputs, $\mathbf{u}_{0:t-1}$

What we don't know is the actual sequence of states that the robot experienced.

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Recursive Bayesian Estimation

The distribution $p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})$ can be complicated.

It may be computed recursively when the state x_t is a sufficient statistic at each t.

In this case, the conditional distribution $p(x_t|\mathbf{x}_{0:t-1}, \mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})$ is equivalent to $p(x_t|x_{t-1}, u_t)$.

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We often solve problems *as if* our system is Markovian, even though we don't really know if that's true

Bayes Filter

Our estimate of the state x at time t is captured by the distribution

$$\mathrm{bel}(x_t) = p(x_t | \mathbf{y}_{1:t}, \mathbf{u}_{1:t}),$$

where we assume this distribution at time t is computed after u_t is taken and y_t measured. This distribution is our *belief* about the state x_t.

Bayes' rule allows us to derive

$$\mathrm{bel}(x_t) = \eta p(y_t | x_t) \int p(x_t | x_{t-1}, u_t) \mathrm{bel}(x_{t-1}) dx_{t-1}$$

Bayes Filter: Derivation

$$
\begin{aligned}
p(x_t|y_t, \mathbf{y}_{1:t-1}, u_{1:t}) &= \frac{p(y_t|x_t, \mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})p(x_t|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})}{p(y_t|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})} \\
&= \frac{p(y_t|x_t)p(x_t|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})}{p(y_t|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t})} \\
&= \eta p(y_t|x_t)p(x_t|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t}) \\
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&= \eta p(y_t|x_t) \int p(x_t|u_t, x_{t-1})p(x_{t-1}|\mathbf{y}_{1:t-1}, \mathbf{u}_{1:t-1})dx \\
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\end{aligned}
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Bayes Filter: Alternate Derivation

We have a model $p(x_t|x_{t-1}, u_t)$. If we know x_{t-1}, u_t, then we could set our motion-based update as $\overline{\mathrm{bel}}(x_t) = p(x_t|x_{t-1}, u_t)$

When we have uncertainty in x_{t-1}, corresponding to $\mathrm{bel}(x_{t-1})$, we instead calculate

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$$\overline{\mathrm{bel}}(x_t) = \int p(x_t|x_{t-1}, u_t)\mathrm{bel}(x_{t-1})dx_{t-1}$$

$\overline{\mathrm{bel}}(x_t)$ is our new belief distribution for x_t We can then apply Bayes' rule to the observation y_t:

$$p(x_t|y_t) = \frac{p(y_t|x_t)p(x_t)}{p(y_t)} = \eta p(y_t|x_t)\overline{\mathrm{bel}}(x_t)$$

Kalman Filter

The Kalman Filter was developed for discrete-time linear time-invariant dynamical systems driven by noise:

$$x_{t+1} = Ax_t + v_t; \quad y_t = Cx_t + w_t,$$

where v_t and w_t represent zero-mean gaussian noise with covariances V and W respectively.

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The Bayesian update

$$\mathrm{bel}(x_t) = \eta p(y_t|x_t) \int p(x_t|x_{t-1}, u_t)\mathrm{bel}(x_{t-1}) dx_{t-1}$$

are easier to handle for this type of system, so that the Kalman Filter is often taught to control theorists without informing them about Bayes' existence.

Kalman Filter

We represent $\mathrm{bel}(x_t)$ using a mean $\mu_t = \mathbb{E}\left[x_t\right]$ and error covariance $\Sigma_t = \mathbb{E}\left[(x_{t+1} - \mu_t)(x_{t+1} - \mu_t)^T\right]$.

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If $x_{t+1}^{pred} = Ax_t + v_t$, then

$$\begin{aligned} \mu_{t+1}^{pred} &= \mathbb{E}[x_{t+1}] = \mathbb{E}[Ax_t + v_t] \\ &= A\mathbb{E}[x_t] + \mathbb{E}[v_t] \\ &= A\mu_t \end{aligned}$$

The predicted error covariance is then

$$\begin{aligned} \Sigma_{t+1}^{pred} &= \mathbb{E}\left[(x_{t+1} - \mu_{t+1}^{pred})(x_{t+1} - \mu_{t+1}^{pred})^T\right] \\ &= \mathbb{E}\left[(Ax_t + v_t - A\mu_t)(Ax_t + v_t - A\mu_t)^T\right] \\ &= A\Sigma_t A^T + V \end{aligned}$$

Kalman Filter

Once we get the measurement y_{t+1}, we then correct the predicted mean as

$$\mu_{t+1} = \mu_{t+1}^{pred} + L(y_{t+1} - C\mu_{t+1}^{pred}) = (I - LC)\mu_{t+1}^{pred} + Ly_{t+1} = M\mu_{t+1}^{pred} + Ly_{t+1},$$

where $M = I - LC$.

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where $M = I - LC$.

Note that $y_{t+1} = Cx_{t+1} + w_{t+1}$. The error covariance at $t+1$ is

$$\begin{aligned}
\Sigma_{t+1} &= \mathbb{E}\left[(x_{t+1} - \mu_{t+1})(x_{t+1} - \mu_{t+1})^T\right] \\
&= \mathbb{E}\left[\left(x_{t+1} - M\mu_{t+1}^{pred} - Ly_{t+1}\right)\left(x_{t+1} - M\mu_{t+1}^{pred} - Ly_{t+1}\right)^T\right] \\
&= \mathbb{E}\left[\left(M(x_{t+1} - \mu_{t+1}^{pred}) - Lw_{t+1}\right)\left(M(x_{t+1} - \mu_{t+1}^{pred}) - Lw_{t+1}\right)^T\right]
\end{aligned}$$

Final Error Covariance

$$
\begin{aligned}
\Sigma_{t+1} &= \mathbb{E}\left[(x_{t+1} - \mu_{t+1})(x_{t+1} - \mu_{t+1})^T\right] \\
&= \mathbb{E}\left[\left(x_{t+1} - M\mu_{t+1}^{pred} - Ly_{t+1}\right)\left(x_{t+1} - M\mu_{t+1}^{pred} - Ly_{t+1}\right)^T\right] \\
&= \mathbb{E}\left[\left(M(x_{t+1} - \mu_{t+1}^{pred})\right)\left(M(x_{t+1} - \mu_{t+1}^{pred})\right)^T\right] \\
&\quad + \mathbb{E}\left[Lw_{t+1}(Lw_{t+1})^T\right] \\
&= \mathbb{E}\left[\left(M(x_{t+1} - \mu_{t+1}^{pred})\right)\left(M(x_{t+1} - \mu_{t+1}^{pred})\right)^T\right] \\
&\quad + \mathbb{E}\left[Lw_{t+1}(Lw_{t+1})^T\right] \\
&= M\mathbb{E}\left[\left((x_{t+1} - \mu_{t+1}^{pred})\right)\left((x_{t+1} - \mu_{t+1}^{pred})\right)^T\right]M^T \\
&\quad + L\mathbb{E}\left[w_{t+1}(w_{t+1})^T\right]L^T \\
&= (I - LC)\left(A\Sigma_t A^T + V\right)(I - LC)^T + LWL^T
\end{aligned}
$$

Kalman Filter Design

The uncertainty in the error depends on the Kalman Gain L.

The optimal choice of L minimizes the expected norm of the error, which is equivalent to minimizing the trace of Σ_{t+1}.

The gradient of the trace of Σ_{t+1} with respect to L becomes

$$-2(C(A\Sigma_t A + V))^T + 2L\left(C\left(A\Sigma_t A + V\right)C^T + W\right),$$

so that the optimal Kalman gain is

$$L = (A\Sigma_t A + V)C^T\left(C\left(A\Sigma_t A + V\right)C^T + W\right)^{-1}$$

At which point the covariance Σ_{t+1} becomes

$$\Sigma_{t+1} = (I - LC)(A\Sigma_t A + V)$$