ODE Models

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1 Introduction: Dynamical Systems

We want to describe the behavior of various physical and non-physical variables over time. To be able to do so, we introduce the notion of signals and systems.

1.1 Signals

A physical variable, for example the speed v of a car, is a well-defined concept independent of time. This physical variable belongs to a set, for example $v \in \mathbb{R}$, the set of real numbers.

We can associate a physical variable with time. For example, the positions of all cars on a road, or currents through components of an electrical circuit. This association takes the form of a function that maps time to the set of values that the variable can take. For the example of a car's speed, we have

$$v\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto v(t).$$

Observe that we often use the same symbol that represents a physical quantity to also represent the physical variable as a function of time. The notation above says that the *function* v is a map from time (the set of real numbers) to the possible values that the *physical variable* can take (which is also the set of real numbers). The notation also shows what the transformation would look like. A given value of time t is mapped to the value of the speed of the car at time t.

These functions of time constitute **signals**. We will see how we can view systems in terms of signals and how they modify signals.

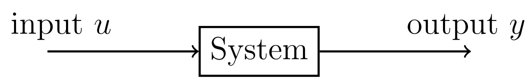
1.2 Systems

A system is some subset of the universe relevant to us. The remainder of the universe becomes the environment with which the system interacts. Note that defining a system becomes an act of choice.

The interaction between a system and its environment occurs through inputs and outputs. An input is a physical quantity that is directly influenced by the environment, and cannot be changed by the system. To most objects on Earth, gravitational force is an input. For a vehicle suspension system, the road surface profile is an input. An output is typically a physical quantity that can be influenced by the system, and also affect the environment. It is not required that an output actually affect the environment. What we can or want to observe about the system is often taken as its output.

Example 1. A bouncing ball is a system that interacts with the rest of the universe through gravitational forces, and the reaction forces with the ground. The height of the ball above the ground is one possible output.

From the view of the environment, the system accepts inputs from the environment and produces outputs.



Assume that we have some meaningful physical variables q that describe a system of interest. For example, the positions of all cars on a road, or currents through components of an electrical circuit. Suppose we know all physical inputs $u(t)$ to the system for a future period of time. For the cars, let the possible inputs be the forces acting on the car, typically the reaction forces at the wheels, gravity, and air drag. For a circuit, the input may be a voltage applied across two points. How do we know what $q(t)$ will be, given $u(t)$?

In the simplest case, $q(t)$ is an algebraic function of the input $u(t)$. For example, the output current $i(t)$ through a resistance R across which we apply a given input voltage $e(t)$ is simply

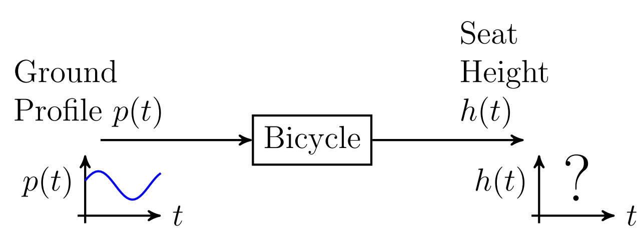
$$i(t) = \frac{1}{R}e(t).$$

For such a system, time is irrelevent, since the output at one moment of time doesn't depend on any other moment of time.

1.3 Dynamical Systems

A dynamical system is one where time factors into the relationship between inputs and outputs.
The inputs are <u>*signals*</u> $u(t)$, and the outputs are also <u>*signals*</u> $y(t)$ A dynamical system can then be thought of a signal that converts one signal into another. This conversion occurs in 'real-time'.

For example, consider a bicycle as the system, with the ground's shape as an input over time and the height of the seat as output. **Why is the output not just a multiple of the input at each time?** The answer is that the system has an internal state which remembers the history of inputs, and the output depends on this history through dependence on the **state** of the system.



Definition 1 (State)**.** The *state* of a dynamical system consists of a set of independent quantities that together allow prediction of the state at future times, given the inputs applied to the system for all future time.

The state summarizes the history of a system, since the current state and (future) input dictates the future state. State is typically denoted by x, and state at time t is $x(t)$. The next section discusses the use of ordinary differential equations (ODEs) to predict the future state from the current state and future inputs.

Sumary. A dynamical system consists of inputs, outputs, and a state. The next section describes the situation where ODEs describe the evolution of the state over time.

2 ODE Models

A **differential equation** is an equation that relates one or more functions and their derivatives. An **ordinary differential equation** (ODE) is a differential equation containing one or more functions of **one** independent variable and the derivatives of those functions. The term ordinary is used in contrast with the term partial differential equation which may be with respect to more than one independent variable. For example consider the following models:

$$\begin{aligned}
\text{ODE:} \quad & \frac{d^2}{dt^2}q(t) + \frac{d}{dt}q(t) + \sin q(t) + (q(t))^3 = 0 \\
\text{ODE:} \quad & \frac{d^2}{dx^2}q(x) + \frac{d}{dx}q(x) + \sin q(x) + (q(x))^3 = 0 \\
\text{PDE:} \quad & \frac{\partial^2}{\partial x^2}q(x,y) + \frac{\partial}{\partial y}q(x,y) + \sin q(x,y) + (q(x,y))^3 = 0
\end{aligned}$$

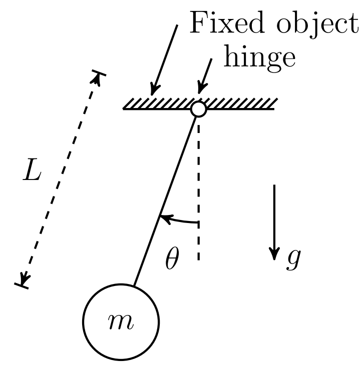
2.1 ODEs From First-Principles

For some systems, the physical quantities of interest (the output signals) are not algebraically related to the input signals. Instead, principles from physics, or another domain, define an algebraic relationship between physical quantities and **their time derivatives**. These relationships define a set of **ordinary differential equations** containing those variables. We refer to these ordinary differential equations as the **equations of motion** (EOMs) of the system. Several systems can be usefully, if not perfectly accurately, modeled by a set of ordinary differential equations.

Example 2 (Simple Pendulum)**.** The simple pendulum is a mass m suspended by a rigid massless string of length L from a point, moving under the effect of gravity g. This system can be described by the angle θ, with time t as the independent variable. The rotational version of Newton's laws provide the EoM:

$$mL^2\ddot{\theta} + mgL\sin\theta = 0,$$

which is a second-order differential equation in $\theta(t)$ with independent variable t.



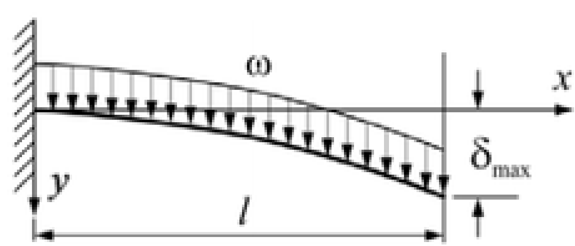
Summary. We focus on systems whose physical quantities all depend on a single one-dimensional variable, usually time or a single spatial dimension (called the independent variable). Once we make this choice, we are dealing with ODE models of the how these quantities change with respect to the independent variable. We refer to the ODEs as the **Equations of Motion.**

2.2 When Are ODE Models Appropriate?

We've seen the idea that models are often wrong, in the sense that they are not perfect. However, these imperfect models are often still useful for making predictions about a system. When are ODE models appropriate? By definition, ODE models require the variables to be **functions of a single independent variable**, which is often time.

The angle of the simple pendulum only depends on time, and therefore we expect an ODE model to be appropriate. Either from experience, or this class, you will know that the simple pendulum tends to oscillate around the downward position ($\theta = 0$).

Consider a cantilever beam on the right. We want to know the vertical displacement at each point on the beam, under different loading conditions. That is, what is the relationship between the shape of the beam and the forces on it? Since the beam is a continuum, this vertical displacement is a (continuous) function of position x along the beam, say $\nu(x)$. If this function changes over time, we represent it instead by $\nu(x, t)$.



The general equation governing the evolution of $\nu(x,t)$ is

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2}{\partial x^2}\nu(x,t)\right) = -\mu\frac{\partial^2}{\partial t^2}\nu(x,t) + q(x), \tag{1}$$

where $E(x)$ is the Young's Modulus along the beam, $I(x)$ is the cross-sectional moment of inertia along x, μ is the linear mass density, and $q(x)$ is the loading force applied at each x.

Equation (1) is a PDE, and is hard to solve in closed-form unless $p(x) = 0$ for each x. This condition corresponds to free vibrations of the beam, which is interesting, but leaves out many other engineering scenarios. For example, what happens if there's a heavy mass m at the end? then $q(l) = mg \neq 0$ and solving this equation is challenging.

Equilibrium behavior and ODEs. In ME 302, you would seek to understand the equilibrium position $\nu(x)$ of the beam. The partial derivative due to time disappears, leaving

$$\frac{d^2}{dx^2}\left(E(x)I(x)\frac{d^2}{dx^2}\nu(x)\right) = q(x) \tag{2}$$

For the case of a uniform beam, $E(x) = E$, $I(x) = I$ for all possible x, so we get

$$EI\frac{d^4}{dx^4}\nu(x) = q(x) \tag{3}$$

This model is an ODE, since there is only one independent variable, x. Common engineering scenarios correspond to specific loading functions $q(x)$, and engineering textbooks contain closed form solutions for the end-point deflection $\nu(l)$ given such $q(x)$, where l is the length of the beam.

Unloaded Free Oscillations and ODEs: When $q(x) = 0$, we may separate the solution $\nu(x,t)$ of (1) into $\nu(x,t) = X(x)f(t)$ (See this webpage for more details). This separation means we can think of a shape $X(x)$ that gets scaled by a number $f(t)$ that changes over time.

Rewriting (1), under the assumption of uniformity, we get

$$EI\frac{\partial^4}{\partial x^4}\left(X(x)f(t)\right) = -\mu\frac{\partial^2}{\partial t^2}\left(X(x)f(t)\right) \tag{4}$$

$$\implies EIf(t)\frac{\partial^4}{\partial x^4}X(x) = -\mu X(x)\frac{\partial^2}{\partial t^2}f(t) \tag{5}$$

$$\implies \frac{EI}{\mu X(x)}\frac{\partial^4}{\partial x^4}X(x) = -\frac{1}{f(t)}\frac{\partial^2}{\partial t^2}f(t) = \omega_n^2 \tag{6}$$

We may replace partial derivatives by total derivatives. The equation determining $f(t)$ is now

$$\frac{d^2}{dt^2}f(t) + \omega_n^2 f(t) = 0.$$

This equation is a second-order differential equation much like the simple pendulum above. The beam truly vibrates. We may think of this solution as a shape $X(x)$ (also called mode) that oscillates with amplitude $f(t)$.

2.3 Lumped vs Distributed Parameter Models

The cantilever beam is challenging to analyze because mass and elasticity are continuously distributed throughout the beam, and especially along the length of it. This continuum nature almost always leads to PDE-based models, which are difficult to handle.

By contrast, models such as that of the Solar system, where bodies are treated as point-masses, are easier to handle. This simplicity due to treating planets as point-masses, instead of bodies with distributed mass, is an example of a common approach to simplifying models. This approach involves assuming that a physical variable spread continuously throughout some region can be replaced by a physical variable of the same kind located at entirely one location, or ascribed to one discrete object. This procedure is known as **lumping** physical variables, and leads to Lumped Parameter Models.

Cantilever. We can also lump the elasticity of a cantilever beam under some conditions. If we apply a single force P at the free end of the beam, $(x = l)$, then the equilibrium deflection is given by $\nu(l) = \frac{PL^3}{3EI}$. Thus, at equilibrium,

$$P = \frac{3EI}{L^3}\nu(l) \text{ which looks like } F = kx.$$

To deflect the end of a (uniform) cantilever beam by y and hold it there, we need a force $(3EI/L^3)y$. This relationship between force on the beam and deflection of the beam is far simpler that what we'd predict from the PDE in (1). More importantly, it suggests that we can view cantilever beams as a linear spring when connected to a mass that's not moving, or **moving slowly**! When the deflection changes slowly, we may lump the distributed elasticity of the beam into a single elasticity given by a linear spring with spring constant $3EI/L^3$. Additionally, we also account for the mass m of the beam by either neglecting it or adding a suitable multiple of it to the mass M attached at the end.

Electrical Circuits. Another example is found in electrical circuits. Every part of the circuit resists the flow of electricity. Resitance is continuously distributed throughout a real circuit. Yet, we assume that all the resitance is located in a finite number of discrete resistors, and all other components offer no resistance to the flow of current. Therefore, we use a lumped physical model to represent what is actually a distributed physical reality.

In many cases, this process is quite acceptable. For example, all the resistance in the circuit may be quite small compared to an individual component, and the lumped parameter model is close-enough to reality. The simplicity of this lumped model is worth the small loss in accuracy.

Not so fast. In some situations, design and analysis using a lumped physical model to represent a distributed reality may lead to wrong and even dangerous designs. One example is in robotic surgery, where it may be dangerous to model the elastic and inertial properties of soft tissue as a lumped system with rigid masses and mass-less springs. Another example is in electrical wiring, where long wires typically thought of as only resistors now act as inductors, affecting the voltages and signals flowing in the electrical system.

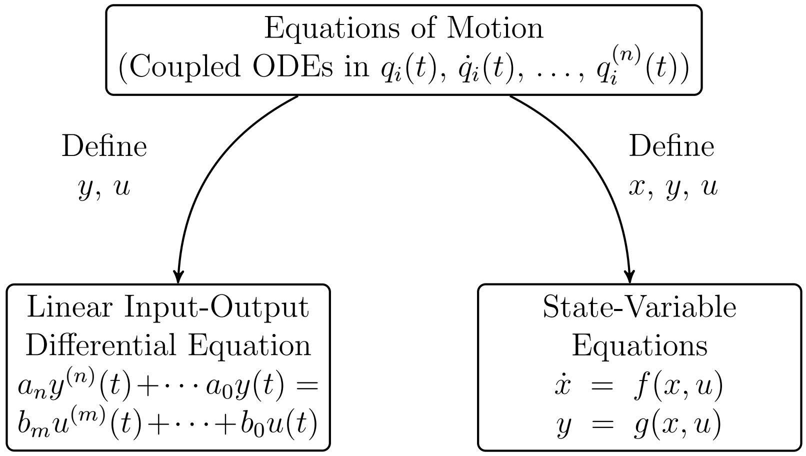


Figure 1: Relationship between ODE representations for a dynamical system.

3 Representations Of ODE Models

We now understand that we will represent dynamical systems, where reasonable, by a set of ordinary differential equations. What will these ODEs look like? There are effectively three kinds of formats for the ODEs:

1. Equations of Motion (EOM)
2. State-Variable Equations (SV)
3. Input-Output Differential Equations (IO)

Figure 1 depicts the relationship between the three representations of dynamical systems as ODES.

3.1 Equations of Motion

The equations of motion are the set of Ordinary Differential Equations obtained after applying domain principles to a system.

Consider the double pendulum in Figure 2. It is made up of two rigid links connected to each other, and one to an immovable reference, using rotational joints. We may derive its equations of motion (EoM) by appying Newton's Second Law to each of the two rigid masses. This procedure would lead to the two differential equations:

$$\begin{aligned}&\left(m_1 L_{c1}^2 + m_2 L_1^2\right)\ddot{\theta}_1 + d(\theta_2)\ddot{\theta}_2 + 2h(\theta_2)\dot{\theta}_1\dot{\theta}_2 + h(\theta_2)\dot{\theta}_2^2 \\ &\quad + m_2 L_2 g\cos(\theta_1+\theta_2) + \left(m_1 L_{c1} + m_2 L_1\right) g\cos\theta_1 = 0\end{aligned} \tag{1a}$$

$$d(\theta_2)\ddot{\theta}_1 + m_2 L_2^2\ddot{\theta}_2 - h(\theta_2)\dot{\theta}_1^2 + m_2 L_2 g\cos(\theta_1+\theta_2) = 0 \tag{1b}$$

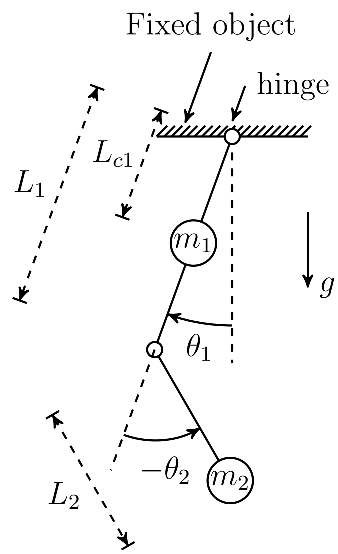


Figure 2: Double Pendulum

where $d(\theta_2) = (m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2)$ and $h(\theta_2) = -m_2 L_1 L_2 \sin\theta_2$. We have neglected the rotational inertia of the links about their centers of masses.

While this representation captures the rules governing the motion of the double pendulum, we aren't able to do much with them. We convert them into either State-Variable Equations, or Input-Output Equations.

3.2 State-Variable Equations

Suppose we know $\dot{q}(t)$ as an explicit function of time, then we can integrate it to get $q(t)$:

$$q(t) = \int_{t_0}^{t} \dot{q}(\tau) d\tau.$$

Now, we have an implicit function in the form of an ODE, that we can solve to obtain $q(t)$.

Suppose we don't have knowledge of $\dot{q}(t)$? Maybe instead we know $\ddot{q}(t)$ as an explicit function of time. Now,

$$\dot{q}(t) = \int_{t_0}^{t} \ddot{q}(\tau) d\tau, \text{ and then } q(t) = \int_{t_0}^{t} \dot{q}(\tau) d\tau.$$

This process is easily extended to knowledge of higher-order derivatives of $q(t)$.

When would we know $q^{(n)}(t) = \frac{d^n}{dt^n} q(t)$ as a function of time? Usually never. Instead, suppose some physical principle provides f such that $\dot{q}(t) = h(q(t), u(t))$. Then,

$$\dot{q}(t) = h(q(t), u(t)) \implies q(t) = \int_{t_0}^{t} \dot{q}(\tau) d\tau = \int_{t_0}^{t} h(q(\tau), u(\tau)) d\tau.$$

Sometimes, we may instead know $\ddot{q}(t)$ as a function of both $\dot{q}(t)$, $q(t)$, and $u(t)$, in the form $\dot{q}(t) = h(\dot{q}(t), q(t), u(t))$. Define the vector

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}.$$

Then

$$\dot{x}(t) = \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} = \begin{bmatrix} \dot{q}(t) \\ h(\dot{q}(t), q(t), u(t)) \end{bmatrix} = f(x(t), u(t)).$$

Again, with some abuse of notation, we have an implicit function for $x(t)$ involving first-order derivatives of $x(t)$ only:

$$x(t) = \int_{t_0}^{t} f(x(\tau), u(\tau)) d\tau.$$

Solving this equation, which is same as solving the ODE $\dot{x}(t) = f(x(t), u(t))$, would yield $x(t)$. If we know that $y(t) = g(x(t), u(t))$, then once we solve for $x(t)$, we know what $y(t)$ is.

The equations

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$
$$y(t) = g(x(t), u(t)) \tag{3}$$

together form the **State-Variable** equations. Notice that

1. The LHS (left-hand side) of (2) is a *first-order* derivative of $x(t)$.
2. The RHS (right-hand side) of (2) is an algebraic function of **only** state $x(t)$ and input $u(t)$.
3. The LHS of (3) is $y(t)$, NOT a derivative of it.
4. The RHS of (3) is also an algebraic function of **only** state $x(t)$ and input $u(t)$.

If the LHS and RHS of these equations do not meet the above condition, then they are not valid SV equations.

3.3 Input-Output Differential Equations

When we use State-Variable Equations, we need to solve as many ODEs are the size of the state $x(t)$. Sometimes, we're interested in a particular function $g(x(t), u(t))$, instead of the full state $x(t)$. In this case, we can define the output $y(t) = g(x(t), u(t))$. However, we avoid the two-step process of solving for $x(t)$ and then using g to compute $y(t)$. Instead, we convert the EOMs into an **Input-Output** Differential Equation and solve for $y(t)$ without ever using the state $x(t)$.

Consider a one-dimensional input $u(t)$ and one-dimensional output $y(t)$. An input-output differential equation for a linear system is then of the form

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) \tag{4}$$
$$= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t) \tag{5}$$

where $a_n, a_{n-1}, \ldots a_0, b_m, b_{m-1}, \ldots, b_0$ are real-valued coefficients. For systems with nonlinear dynamics, some terms in the equation above will be nonlinear.

3.3.1 EoM To IO Equations Using The p-operator

The Equations of Motion are a set of noninear coupled equations with a number of variables. The input-output equations are derived from these equations by a process of elimination identical to solving algebraic systems of equations.

How do we go from differential equations to a system of algebrai equations? We use the p-operator. The idea is

$$\text{Replace } \frac{d}{dt} \to p \tag{6}$$
$$\text{so, } \frac{d}{dt} x(t) \to px \tag{7}$$
$$\frac{d}{dt}\left(\frac{d}{dt} x(t)\right) \to \frac{d}{dt}\left(px(t)\right) \to ppx \to p^2 x \tag{8}$$
$$x^{(n)}(t) = \frac{d^n}{dt^n} x(t) \to p^n x \tag{9}$$

This step is purely for convenience when manipulating differential equations, by replacing the derivative with the quantity p, and treating it as a variable that we manipulate using usual algebra.

Note that

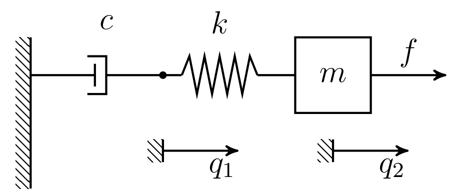
$$p(xy) = ypx + xpy,$$

since

$$\frac{d}{dt}(x(t)y(t)) = y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}$$

Example. Consider the translational mechanical system to the right. We may derive its equations of motion as:

$$m\ddot{q}_2 + kq_2 - kq_1 = f \tag{10}$$
$$c\dot{q}_1 + kq_1 - kq_2 = 0 \tag{11}$$



If we choose the output as $y = q_1$ and input as $u = f$, we get the IO differential equation

$$mcy^{(3)}(t) + mk\ddot{y}(t) + kc\dot{y}(t) = ku(t) \tag{12}$$

To derive the IO equation in (12), we will use the p-operator. First, replace variables with u and y where appropriate.

$$m\ddot{q}_2 + kq_2 - kq_1 = f \tag{13}$$
$$c\dot{q}_1 + kq_1 - kq_2 = 0 \tag{14}$$

becomes

$$m\ddot{q}_2 + kq_2 - ky = u \tag{15}$$
$$c\dot{y} + ky - kq_2 = 0 \tag{16}$$

Now, we replace derivatives using the p-operator:

$$mp^2q_2 + kq_2 - ky = u \tag{17}$$
$$cpy + ky - kq_2 = 0 \tag{18}$$

Collect like terms, treating p as a variable, not a derivative:

$$(mp^2 + k)q_2 - ky = u \tag{19}$$
$$(cp + k)y - kq_2 = 0 \tag{20}$$

Now, we just need to eliminate q_2 so that we get an equation in just y and u, **and their derivatives** as captured by p. Note that other problems may require us to eliminate more than just one variable.

Solve for q_2 in (20):

$$(cp + k)y - kq_2 = 0 \implies q_2 = \frac{cp + k}{k}y \tag{21}$$

Substitute this into (19):

$$(mp^2 + k)\underbrace{q_2}_{\text{replace}} - ky = u \to \underbrace{(mp^2 + k)\frac{(cp + k)}{k}y - ky}_{\text{collect}} = u \tag{22}$$

$$\implies \left(\frac{(mp^2 + k)(cp + k)}{k} - k\right)y = u \tag{23}$$

$$\implies \left(\frac{(mcp^3 + mkp^2 + kcp + k^2 - k^2)}{k}\right)y = u \tag{24}$$

$$\implies (mcp^3 + mkp^2 + kcp)y = ku \tag{25}$$

$$\implies mcp^3y + mkp^2y + kcpy = ku \tag{26}$$

$$\implies mcy^{(3)}(t) + mk\ddot{y}(t) + kc\dot{y}(t) = ku(t) \tag{27}$$

In the last step, we replace the p-operator by derivatives applied to the variable on the right.

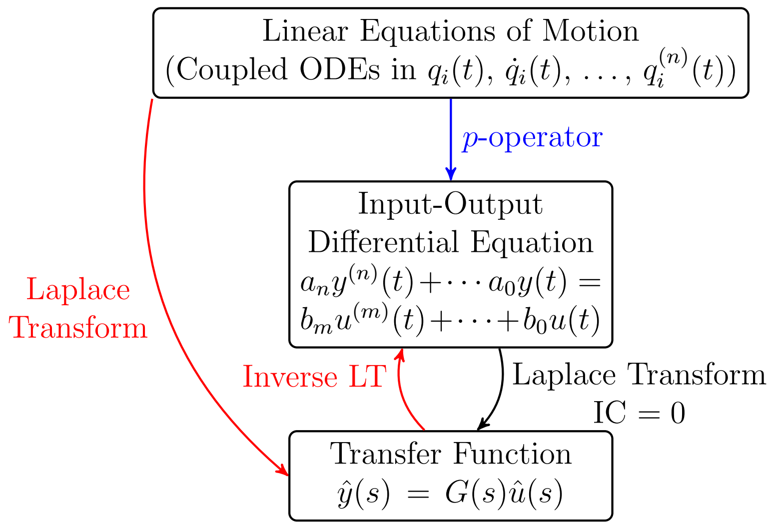


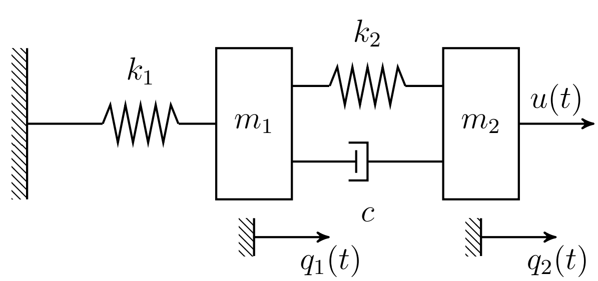
Figure 3: From EOMs to IO to Transfer Functions. Instead of using the p-operator approach, we may derive IO equations from IOM by computing a Laplace Transform and then its inverse.

3.3.2 Relationship to Laplace Transforms

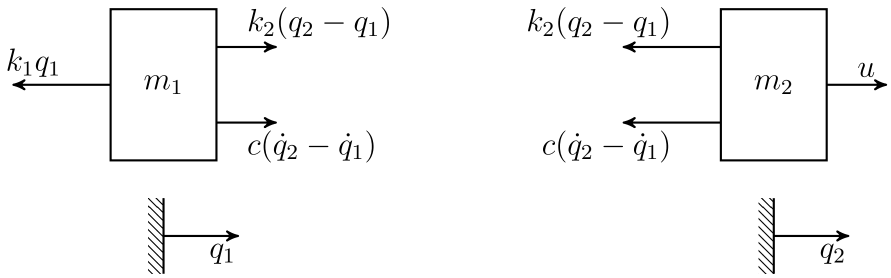
Later, we will see how to use Laplace transforms to obtain transfer functions that relate input to ouputs. These Transfer functions are closely related to IO equations. We can easily convert Transfer functions into IO differential equations and back. Figure 3 depicts this seemingly round-about approach.

4 Examples

Problem 1 (Textbook Example 3.2)**.** Write output in terms of spring tension f_{s_2} and total momentum m_T of masses.



Solution: We get the free-body diagrams:



Applying Newton's second law, we get EOMs:

$$m_1\ddot{q}_1 = k_2(q_2 - q_1) + c(\dot{q}_2 - \dot{q}_1) - k_1 q_1, \text{ and} \tag{1}$$

$$m_2\ddot{q}_2 = u - k_2(q_2 - q_1) - c(\dot{q}_2 - \dot{q}_1). \tag{2}$$

How should we choose the state?

1. By default: use positions q_1, q_2, and their velocities $v_1 = \dot{q}_1$, $v_2 = \dot{q}_2$. Later, remove non-independent states.
2. Determine which physical variables are needed to compute the outputs, and use them to define a state.

The solution in the textbook goes with the first approach, arguing that the default choice results in independent states.

Default approach:
The state is taken to include q_1, q_2, v_1, and v_2. As usual, we get the first two equations directly, and the derivatives $\dot{v}_1$ and $\dot{v}_2$ from (1) and (2):

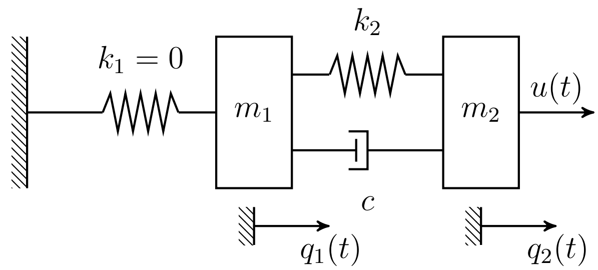
$$\begin{aligned}
\dot{q}_1 &= v_1, \\
\dot{q}_2 &= v_2, \\
\dot{v}_1 &= \frac{1}{m_1}\left(k_2(q_2 - q_1) + c(v_2 - v_1) - k_1 q_1\right), \text{ and} \\
\dot{v}_2 &= \frac{1}{m_2}\left(u - k_2(q_2 - q_1) - c(v_1 - v_1)\right).
\end{aligned}$$

Outputs:

$$\begin{aligned}
y_1 &= k_2(q_2 - q_1), \text{ and} \\
y_2 &= m_1 v_1 + m_2 v_2.
\end{aligned}$$

There are no independent states, and so these are the state-variable equations.

Output approach: The outputs are $y_1 = k_2(q_2 - q_1)$, and $y_2 = m_1 v_1 + m_2 v_2$. Clearly, the state will need to include q_1, q_2, v_1, and v_2 for the outputs to be an algebraic function of the state.



Problem 2 (Textbook Example 3.5)**.** Write output in terms of spring tension f_{s_2} and total momentum m_T of masses, when $k_1 = 0$. Repeat the problem when q_1 is an additional output of interest.

Solution:
The textbook states that we should expect three state variables because there are three energy storing elements: two masses and a spring. The velocities v_1 and v_2 are chosen as states, as is the relative displacement $q_R = q_2 - q_1$.

Default approach:
Setting $k_1 = 0$ in the default solution of Example 3.2 results in:

$$\begin{aligned}
\dot{q}_1 &= v_1, \\
\dot{q}_2 &= v_2, \\
\dot{v}_1 &= \frac{1}{m_1}\left(k_2(q_2 - q_1) + c(v_2 - v_1)\right), \text{ and} \\
\dot{v}_2 &= \frac{1}{m_2}\left(u - k_2(q_2 - q_1) - c(v_1 - v_1)\right).
\end{aligned}$$

There's no clear way to determine algebraic relationships between the state variables that enable us to eliminate dependent state variables.

Ouput approach:
If the output is just $f_{s_2} = k_2(q_2 - q_1)$, let's choose one state as $q_R = q_2 - q_1$. We don't have a method to calculate $\dot{q}_R(= v_R)$. We **have** a rule to calculate $\ddot{q}_R$ $(= \dot{v}_R)$. Therefore, by including v_R in the state, we can write the equations

$$\begin{aligned}
\dot{q}_R &= v_R, \\
\dot{v}_R &= \ddot{q}_2 - \ddot{q}_1 \\
&= -\frac{1}{m_1}\left(k_2(q_2 - q_1) + c(v_2 - v_1)\right) + \frac{1}{m_2}\left(u - k_2(q_2 - q_1) - c(v_1 - v_1)\right) \\
&= \frac{1}{m_2}\left(u - k_2 q_R - c v_R\right) - \frac{1}{m_1}\left(k_2 q_R + c v_R\right) \\
y &= k_2 q_R
\end{aligned}$$

which is a valid state-variable representation of the system.

If the output m_T is included, we would have $y_2 = m_1 v_1 + m_2 v_2$. There's no way to make m_T an algebraic function of q_R and v_R. This issue suggests that we need to include either v_1 or v_2 in the state:

$$\begin{aligned}
\dot{q}_R &= v_R, \\
\dot{v}_R &= -\frac{1}{m_1}(k_2 q_R + c v_R) + \frac{1}{m_2}(u - k_2 q_R - c v_R) \\
\dot{v}_2 &= \frac{1}{m_2}(u - k_2 q_R - c v_R) \\
y_1 &= k_2 q_R \\
y_2 &= m_1(v_2 - v_R) + m_2 v_2
\end{aligned}$$

Instead, one can choose to use v_1:

$$\begin{aligned}
\dot{q}_R &= v_R, \\
\dot{v}_R &= -\frac{1}{m_1}(k_2 q_R + c v_R) + \frac{1}{m_2}(u - k_2 q_R - c v_R) \\
\dot{v}_1 &= \frac{1}{m_1}(k_2 q_R + c v_R) \\
y_1 &= k_2 q_R \\
y_2 &= m_1 v_1 + m_2(v_R + v_1)
\end{aligned}$$

Instead, we can replace v_R by v_1 and v_2:

$$\begin{aligned}
\dot{q}_R &= v_2 - v_1, \\
\dot{v}_1 &= \frac{1}{m_1}(k_2 q_R + c(v_2 - v_1)) \\
\dot{v}_2 &= \frac{1}{m_2}(u - k_2 q_R - c(v_2 - v_1)) \\
y_1 &= k_2 q_R \\
y_2 &= m_1 v_1 + m_2 v_2
\end{aligned}$$

If additionally an output is taken to be q_1, the same problem arises. We can't represent the output q_1 in terms of the states we have chosen so far. The correct solution is to add q_1 as a state

$$\begin{aligned}
\dot{q}_1 &= v_1 \\
\dot{q}_R &= v_2 - v_1, \\
\dot{v}_1 &= \frac{1}{m_1}\left(k_2 q_R + c(v_2 - v_1)\right) \\
\dot{v}_2 &= \frac{1}{m_2}\left(u - k_2 q_R - c(v_2 - v_1)\right) \\
y_1 &= k_2 q_R \\
y_2 &= m_1 v_1 + m_2 v_2 \\
y_3 &= q_1
\end{aligned}$$

Example 3. Consider a dynamic system modeled by

$$\ddot{q}_1 + 2\dot{q}_2 + 3q_1 = f \tag{3}$$
$$\ddot{q}_2 + 4\dot{q}_1 + 5q_2 = 0 \tag{4}$$

where f is an external force. Find an IO OdE with $u = f$, $y = q_1$.

Plug in $u = f$, $y = q_1$:

$$\begin{aligned}
\ddot{y} + 2\dot{q}_2 + 3y &= u \\
\ddot{q}_2 + 4\dot{y} + 5q_2 &= 0
\end{aligned}$$

Rewrite using p operator ($\frac{d^n}{dt^n} q(t) \to p^n q$)

$$\begin{aligned}
p^2 y + 2pq_2 + 3y &= u \\
p^2 q_2 + 4py + 5q_2 &= 0
\end{aligned}$$

Group terms, where p follows normal algebra of numbers:

$$\begin{aligned}
(p^2 + 3)y + 2pq_2 &= u \\
(p^2 + 5)q_2 + 4py &= 0
\end{aligned}$$

Eliminate q_2, leaving y, u, p and model parameters if any:

$$p^4 y + 15y = p^2 u + 5u.$$

This equation implies that

$$y^{(4)}(t) + 15y(t) = \ddot{u}(t) + 5u(t).$$