Mechanical Systems

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Contents

1 Introduction

We view a mechanical system as a collection of objects that have four major types of forces:

1. Inertial forces
2. Elastic forces
3. Frictional forces
4. Reaction forces

Inertial forces are forces that are experienced when the momentum of an object changes. Elastic forces arise due to a resistance to change in shape. Frictional forces arise due to a resistance to relative motion between surfaces in contact. Reaction forces typically arise at contacts, and represent constraints of motion.

1.1 Lumped Models

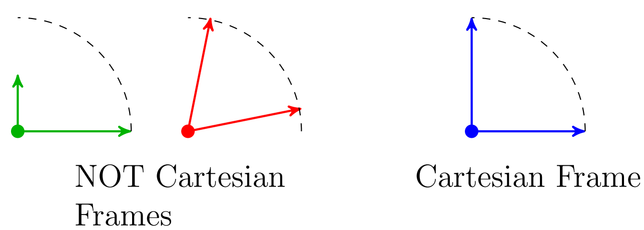
An object like a physical spring is made material that has both mass and elasticity. Similarly, a block of metal will compress slightly when appropriate forces are applied. However, for purposes of modeling, we will assume that springs are massless, and objects with masses lack elasticity, that it is, they are perfectly rigid. We are effectively lumping the mass and elasticity in a system into idealized masses and idealized springs.

1.2 Frames of Reference and Motion

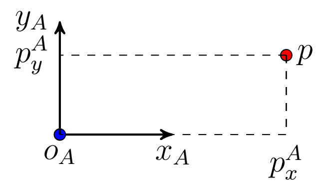
This course focuses on systems with two spatial dimensions. Therefore, we can draw all our systems on a sheet of paper, representing the two-dimensional plane. Most of the content can be extended to the three-dimensional space directly. The symbol $\mathbb{R}^2$ denotes this plane.

Points in the two-dimensional plane do not have intrinsic coordinates. Every Cartesian coordinate frame assigns its own unique coordinate to a point in two-dimensions.

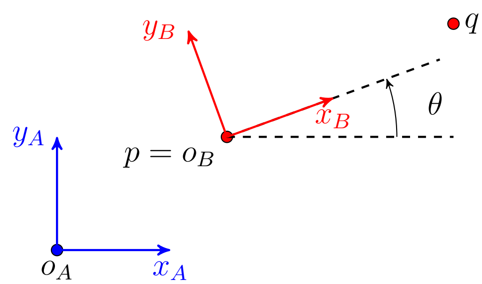
A Cartesian coordinate frame in $\mathbb{R}^2$ consists of a single point and two vectors. These two vectors must be perpendicular, and have the same length. The vectors effectively *define* what unit length means in this coordinate system. Moreover, in a Cartesian frame, unit length doesn't change when you rotate objects.



Consider a Cartesian coordinate frame A with origin o_A and vectors x_A and y_A that define the $x-$ and $y-$ axis of frame A. The Cartesian coordinates (p_x^A, p_y^A) of a point p in frame A are the lengths between the origin and the points of intersections of perpendicular lines from q to the $x-$ and $y-$ axes.



The same point in space can have multiple coordinates, each corresponding to a different frame. We can relate descriptions of the same point in space in different coordinate frames via rigid coordinate transformations. In the figure to the right, if we are given coordinates of q in frame B, which are (q_x^B, q_y^B), then we compute the coordinates of q in frame A as



$$q^A = \begin{bmatrix} q_x^A \\ q_y^A \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} q_x^B \\ q_y^B \end{bmatrix}}_{\text{Rotation}} + \underbrace{\begin{bmatrix} p_x^A \\ p_y^A \end{bmatrix}}_{\text{translation}} = Rq^B + T \tag{1}$$

This transformation can be split into two terms, one that rotates the original coordinates through the matrix R and then translates by the vector T. Therefore, when describing motion of objects in two dimensions (and also three dimensions), we may **separate** it into translation and rotation.

Main Takeaway. To describe the position of a point, we need to choose a coordinate frame. Since we may define several coordinate frames in the plane, one challenge is making sure that we perform calculations in the **same reference frame**. Relative positions, and therefore motions, may be decomposed into **translation** and **rotation**.

2 Inertia (Masses)

Since motion may be separated into rotational and translational components, we separate inertial forces into translational and rotational components.

2.1 Translation and Translational Inertia

We denote the linear inertial of a body by m. A linear inertia is always associated with a point, and for a body, we associate the entire mass of the body with its center of mass. The center of mass is the average location of all the mass of the body, relative to a reference frame. The center of mass turns out to be independent of the reference frame with respect to which we calculate it, making things easier.

For geometrically symmetric objects with uniform material density, the center of mass is located as the geometric center of the shape. For example, the mass shown as a square on the right has its center of mass at the center of this square. We therefore assign the horizontal location of the mass m as q, relative to the reference indicated by the hashed lines. The horizontal location of mass m, is therefore q. If we associate the position with time, to obtain $q(t)$, we may define its velocity $\dot{q}(t) = dq/dt$ and acceleration $\ddot{q}(t) = d^2q(t)/dt^2$.

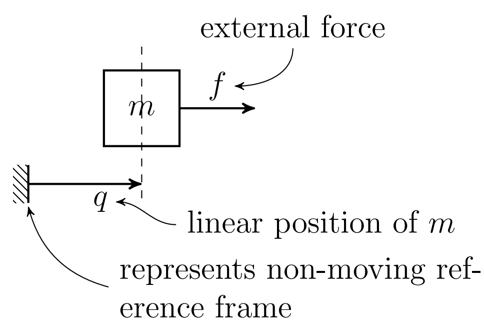


Figure 1: Translating Mass

Convention. For the figure on the right, the value of $q(t)$ is zero. If we matched the method shown in Figure 1, we would be drawing an arrow of length zero, meaning no arrow. Instead, the diagram still contains an arrow labeled q, where $\hat{q}$ indicates the positive sense of the axis that determines the location $q(t)$ of the mass m. In practice, if the arrowhead does not touch a perpendicular line (dashed or solid), then it indicates direction, and its length is unimportant.

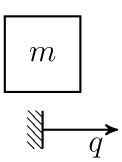


Figure 2: Diagram when $q(t) = 0$

2.2 Rotation and Rotational Inertia

A body is rotating relative to a frame if we can find one point on that body that is moving with respect to that frame, and another point that is stationary with respect to that frame. Often, this stationary point is fixed, for example a disk spinning about an axis on the right.

The angular position is $\theta(t)$ in rad. The angular velocity is $\dot{\theta}(t)$ in rad/s. The angular acceleration is $\ddot{\theta}(t)$ in rad/s^2.

A rotational inertia of an object is associated with an axis. The rotational inertia J of a body that moves in a plane is associated with an axis passing through a point and perpendicular to the plane. Therefore, for two-dimensional systems, the rotational inertia is effectively associated with a point. Again, we typically use the center of mass as a reference point. Unlike for translational inertia, if we use a different point, we get a different value for the rotational inertia (see Parallel Axis Theorem below).

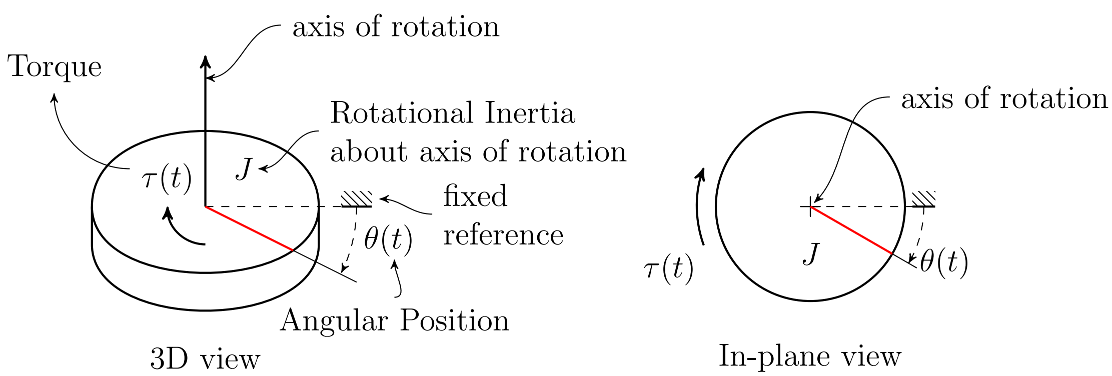
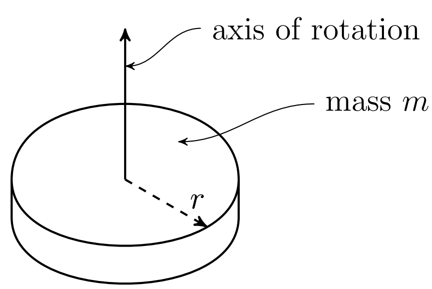


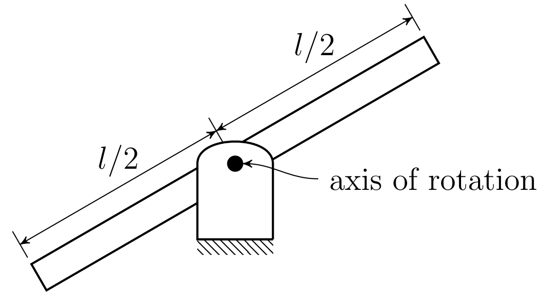
Figure 3: Rotating Mass

For rotational mechanical systems, the (rotational) inertia J depends on the mass (linear inertia) *and* the shape of the body.

Given a disk of mass m and radius r, with uniform thickness, its inertia is $J_{disk} = \frac{1}{2}mr^2$.



Given a bar of mass m and length l, with uniform thickness, its inertia is $J_{bar} = \frac{1}{12}ml^2$.

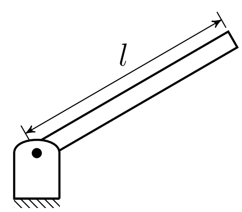


Parallel Axis Theorem. Given a rotational inertia J_0 about an axis passing through the center of mass of an object, the rotational inertia J about an axis parallel to the first axis is given by

$$J = J_0 + ml^2,$$

where l is the distance between the parallel axes.

Example. Given a bar of mass m and length l, with uniform thickness, its inertia about the center of rotation is
$J = J_0 + m\left(\frac{l}{2}\right)^2$, or
$J = \frac{ml^2}{12} + \frac{ml^2}{4} = \frac{ml^2}{3}$



2.2.1 Newton's Laws

Newton's second law gives

$$J\ddot{\theta}(t) = \tau(t).$$

The rotational kinetic energy of this body is

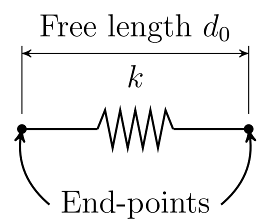
$$\frac{1}{2}J\dot{\theta}^2(t).$$

The power consumed by the body is $\tau(t)\dot{\theta}(t)$ $\mathrm{N \cdot m/s}$.

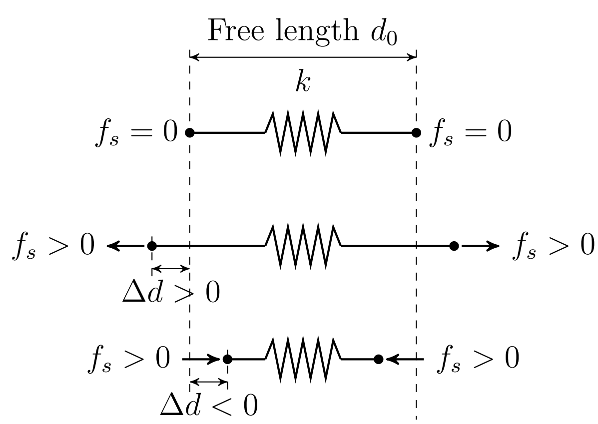
3 Elasticity (Springs)

3.1 Linear Springs

We depict a linear spring using the symbol to the right, which can be described as a zig-zag pattern that mimics a coiled spring. As shown, no forces act on the spring, and the length between its endpoints is called the free length d_0. We assume that this spring is ideal, so that it has no mass. Additionally, this ideal behavior implies that the forces at the two ends are always equal and opposite.



The length of a spring changes based on the force applied at its ends. To calculate the spring force f_s we will use in our models, we need to understand three things:



1. The free length d_0 of the spring
2. The relationship between the extension Δd of the spring and the configuration of the system.
3. The relationship between force f_s and the extension of the spring (spring model); the latter is given by the actual length minus the free length.

If this information is not given, we may assume the following defaults:

1. Default free length: the free length corresponds to the length of the spring when all variables are at 0.
2. Default relationship: The spring force is linear in the spring extension $f_s = k\Delta d$.

Consider the spring on the right, with spring constant k. Its two ends are at configurations $q_1(t)$ and $q_2(t)$. Therefore, the length $d(t)$ of the spring is $d(t) = q_2(t) - q_1(t) + L$. As shown, $q_1(t) = 0$ and $q_2(t) = 0$, so $d(t) = L$. If no other information is given, we may apply the defaults: the free length is the length when all configurations are zero.

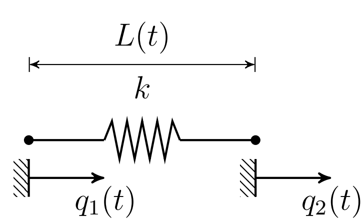


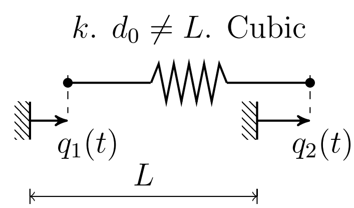
Figure 4:

This means $d_0 = L$. The force-extension relationship is taken to be linear by default. Therefore,

$$
\begin{aligned}
f_s &= k\Delta d && (2)\\
&= k(d(t) - d_0) && (3)\\
&= k(q_2(t) - q_1(t) + L - L) && (4)\\
&= k(q_2(t) - q_1(t)) && (5)
\end{aligned}
$$

NOTE: We could write down the **same expression** for f_s **if** L **was not provided**, by observing that L always cancels out under the default assumptions.

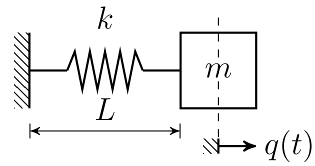
A more complete picture is given to the right. Now, $q_1(t) \neq 0$ and $q_2(t) \neq 0$. The free length has been given as $d_0 \neq L$. In this case,



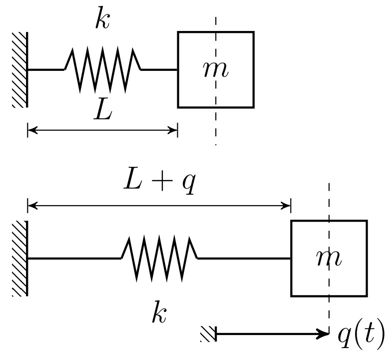
$$
\begin{aligned}
f_s &= k(\Delta d)^3 && (6)\\
&= k(d(t) - d_0)^3 && (7)\\
&= k(q_2(t) - q_1(t) + L - d_0)^3 && (8)
\end{aligned}
$$

which cannot be simplified as before.

3.2 Spring Forces on Inertias

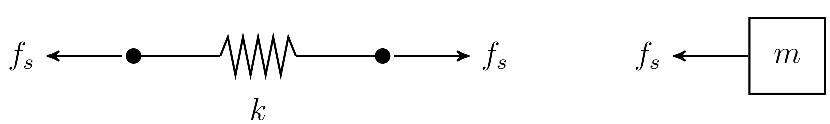


The figure on the right depicts a spring k connected to a fixed object on its left and a mass m on its right. Through these connections, the extension of the spring depends on q, which is the position of the mass m relative to the second reference. As described in Section 3.1, this diagram indicates that the free length of the spring is L (position of mass m is $q = 0$).



Now, we move the mass m to a position $q \neq 0$. In this case, we can draw an arrow from the reference to the actual position of m, unlike the diagram above where q was zero.

By Newton's Third Law, every force has an equal and opposite reaction. So, if a spring is connected to a mass, the motion of the mass may extend or compress the spring. The force applied by the mass on the spring is equal and opposite to the force applied by the spring on the mass. Therefore, the spring force f_s *acting on a mass* must be equal to the force required to produce the extension (or compression) of the spring. This situation is depicted as:

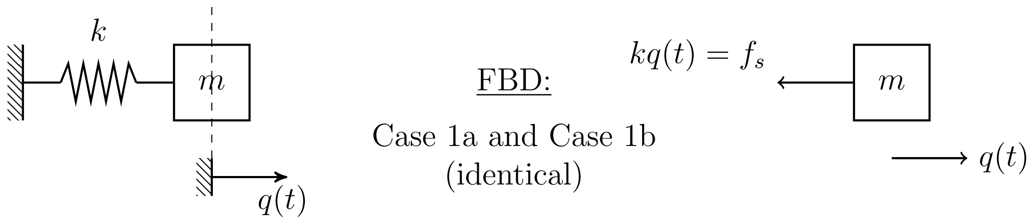


For the relationship between the length of the spring and the configuration q_i of the mass m_i, the critical question is: **does an increase in q_i extend or compress the spring?** There are two ways to use the answer of that question.

1. The FBD always has the spring force pulling the mass m_i towards the spring, and the spring force is proportional to changes that extend the spring. The spring force is proportional to $q_i(t)$ if increasing $q_i(t)$ extends the spring, otherwise it is proportional to $-q_i(t)$. The latter statement is the same as saying increasing $-q_i(t)$ extends the spring, so the spring force is proportional to $-q_i(t)$.

2. The spring force is always proportional to $q_i(t)$, the spring force pulls mass m_i towards it if increasing $q_i(t)$ extends the spring, otherwise it pushes m_i away from it.

Below are some cases where you can see how to apply these two conventions. Use whichever you find easier to apply consistently.



Case 1a: spring to left of mass, $q(t)$ increases to the right. We automatically declare that the FBD of the mass will have f_s on the left pointing leftwards. Increasing $q(t)$ extends the spring, so

$$f_s = kq(t),$$

and $m\ddot{q} = -f_s = -kq(t)$.

Case 1b (Alternative): The spring force is declared to be $f_s = kq(t)$. Since Since increasing $q(t)$ extends the spring, the FBD of the mass will have f_s on the left pointing leftwards (pulling the mass). We use the FBD to find that $m\ddot{q} = -f_s = -kq(t)$.

Note that Case 1a and 1b aren't really distinguishible. The next situation shows where applying the two methods produces different terms, but the same final EOM.

Case 2a: spring to right of mass, $q(t)$ increases to the right. The FBD of the mass will have f_s on the right pointing rightwards. Increasing $q(t)$ compresses the spring, so

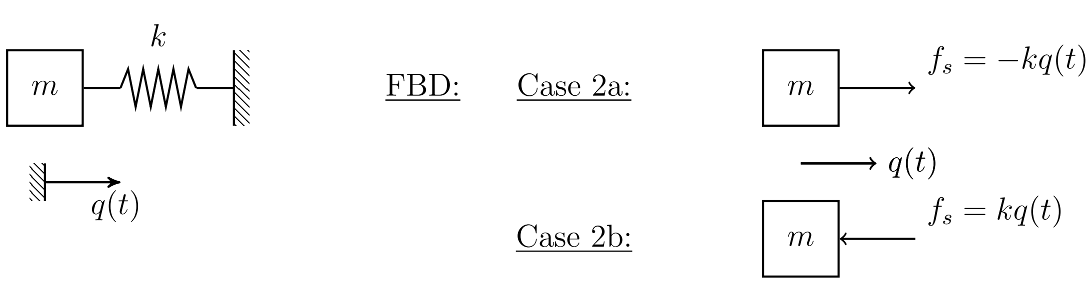
$$f_s = -kq(t),$$

and $m\ddot{q} = f_s = -kq(t)$.

Case 2b (Alternative): spring to right of mass, $q(t)$ increases to the right. We take the spring force to be

$$f_s = kq(t).$$

Increasing $q(t)$ compresses the spring, so the FBD of the mass will have f_s on the right pointing leftwards towards m. Now, $m\ddot{q} = -f_s = -kq(t)$.



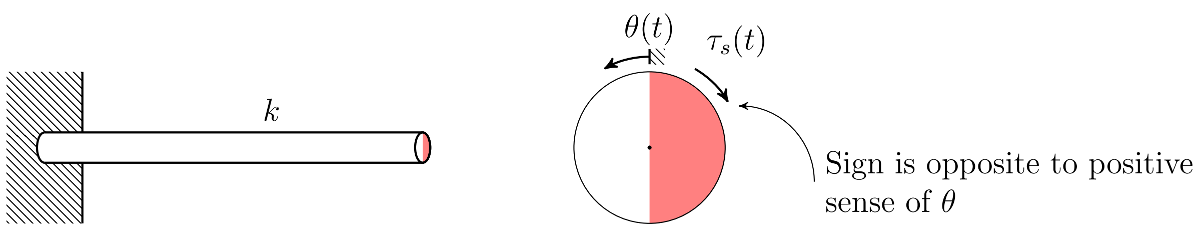
Notice that case $2a$ and $2b$ result in the same dynamics, with different expressions for the spring force.

3.3 Torsional Springs

The rotational stiffness k of an element produces a **restoring** torque τ_s that is an algebraic function of the angular displacement θ of the element. Springs that produce such a spring are called torsional springs. When the algebraic function is linear,

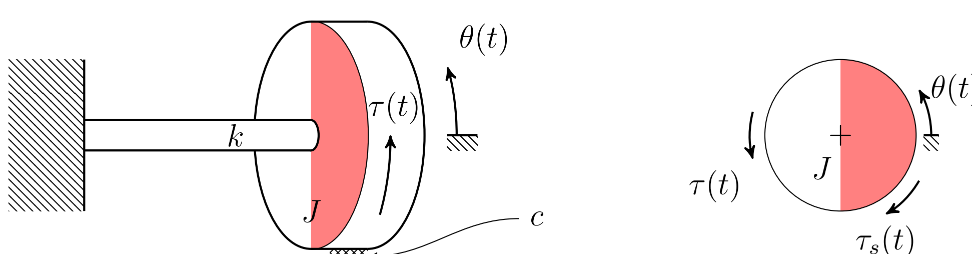
$$\tau_s(t) = k\theta(t),$$

where here the unit of stiffness k is $\mathrm{N \cdot m/rad}$. Since τ_s is a restoring force, its direction will be **opposite** to the direction of **increasing** $\theta(t)$.



Right end of element

Example 1 (Single DOF Rotational System)**.** Write down the equations of motion of the following rotating body:



Free body Diagram

□

Note that the diagram just shows two cylinders. One is long with small radius with symbol k, the other is thin with large radius and symbol J. We therefore interpret the long cylinder as a *massless* torsional spring with rotational stiffness k, and the cylinder J as a perfectly rigid object with rotational inertia J. Again, this model is lumping inertia and elasticity. We assume that J purely rotates, so its translational inertia (mass) is irrelevant.

Apply Newton's second law to get

$$
\begin{aligned}
J\ddot{\theta}(t) &= \tau(t) - \tau_s(t) && (9)\\
&= \tau(t) - k\theta(t) && (10)\\
\implies J\ddot{\theta}(t) + k\theta(t) &= \tau(t) && (11)
\end{aligned}
$$

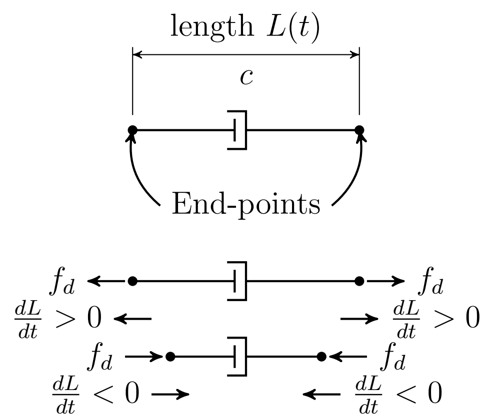
4 Friction (Dampers)

4.1 Linear Friction

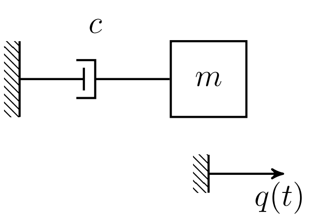
We use a dashpot, or damper, to represent frictional forces. The symbol for a damper is shown in the figure to the right. When no information is provided, we take the model for the force due to this damper to be linear. That is,

$$f_d = c\frac{d}{dt}L(t).$$

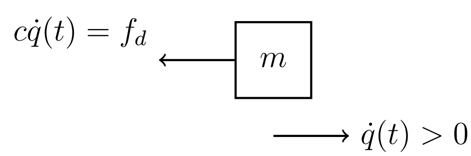
Similar to an ideal spring, an ideal damper has no mass. This assumption means that the damping force at both ends are equal in the ideal case.



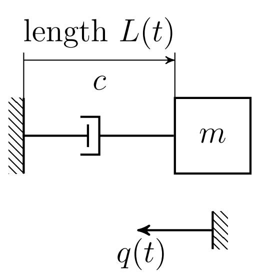
When we connect a damper **to a mass**, the mass feels the opposite force. The diagrams below shows how to correctly account for the force on a masses Free Body Diagram due to a damper:



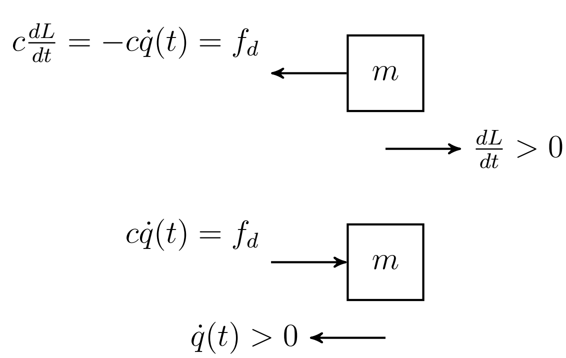
FBD:



When increasing $q(t)$ decreases $L(t)$, we may use two equivalent approaches. One approach is to identify the positive direction of $\dot{L}(t)$, and use a force with magnitude $f_d = c\dot{L}(t)$ and direction such that it points from the mass towards the damper. The other method is to simply apply the idea that the damping force opposes motion, and so the damping force $c\dot{q}(t)$ points in the opposite direction of the mass' velocity.

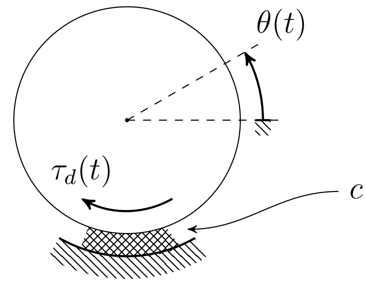


FBD:



4.2 Rotational Friction

The relative motion between two objects may produce a resistive force f_d which is usually modeled as an algebraic function of the linear velocity between the two surfaces.



If an object is in pure rotational motion, the linear velocity is proportional to the angular velocity of the object, and the resistive force acts as a resistive torque τ_d about the axis of rotation.

When this relationship is linear, we can express it as

$$\tau_d = c\dot{\theta}(t),$$

where c has units $\mathrm{N \cdot m \cdot s/rad}$ Here, the simplest approach is to observe that the damping torque has direction opposite to the positive sense of θ, which is also the positive sense of $\dot{\theta}(t)$.

5 Connectors and Pulleys

Our model of a system will combine lumped inertias, springs, and frictional elements. Some of these elements only translate, some will only rotate about a fixed axis, and some elements will both translate and rotate.

In keeping with our approach of lumping physical quantities into representative elements, we represent physical connections between elements by the following types of elements:

1. Perfectly rigid massless rods:
   - Force on both ends are always equal (like spring, damper)
   - Length never changes $\implies$ velocity of both ends are equal
2. Perfectly inextensible massless strings
   - Tension acts along string, and is equal at all points
   - When tension positive, velocity of both ends are equal
   - The tension cannot be negative. Physically, the string is 'slack' and applies zero force on the objects it is connected to. The motion of the ends are independent.

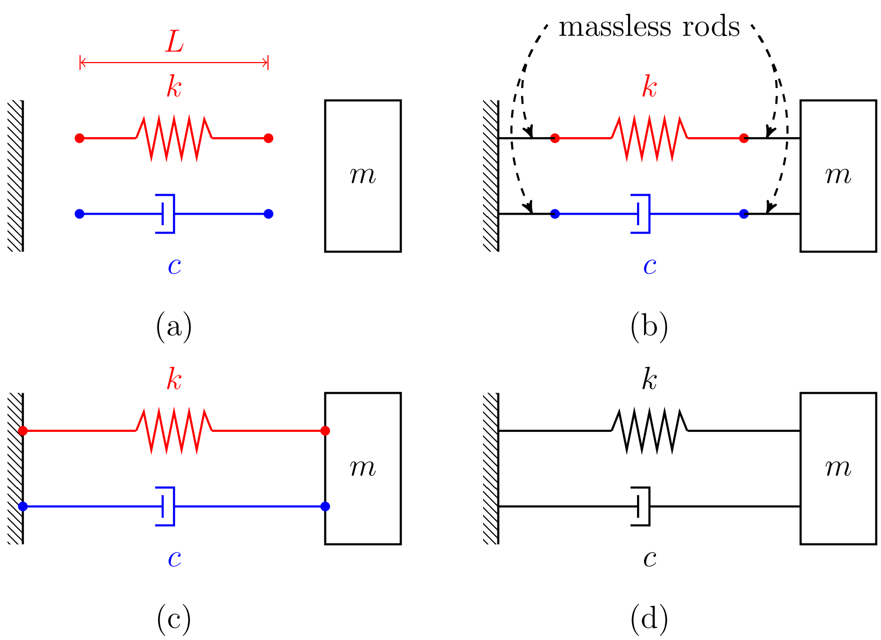


Figure 5: The use of massless rods to connect the mass, spring, and damper lead to a simplified diagram in Figure 5d in which the ends of the elements are not shown. Figure 5a-c shows that the massless rods effectively move the end-points of the elements, since the same force is transmitted at all points. This diagram also explains why we prefer to infer the free-length of the spring from the diagram and positions of masses, since the end-points can be moved on mass-less springs without changing the extension of the spring.

Figure 5 shows an example of how we connect elements using massless rods, and then simplify the diagram by removing explicit points of connection. Similarly, Figure 6 depicts connection between elements using strings.

An important idea is that these connectors are ideal, and so we never observe any forces due to those connectors. The tension T is an internal force, and **should never appear in a final set of Equations of Motion**, although it shows up when applying Newton's laws.

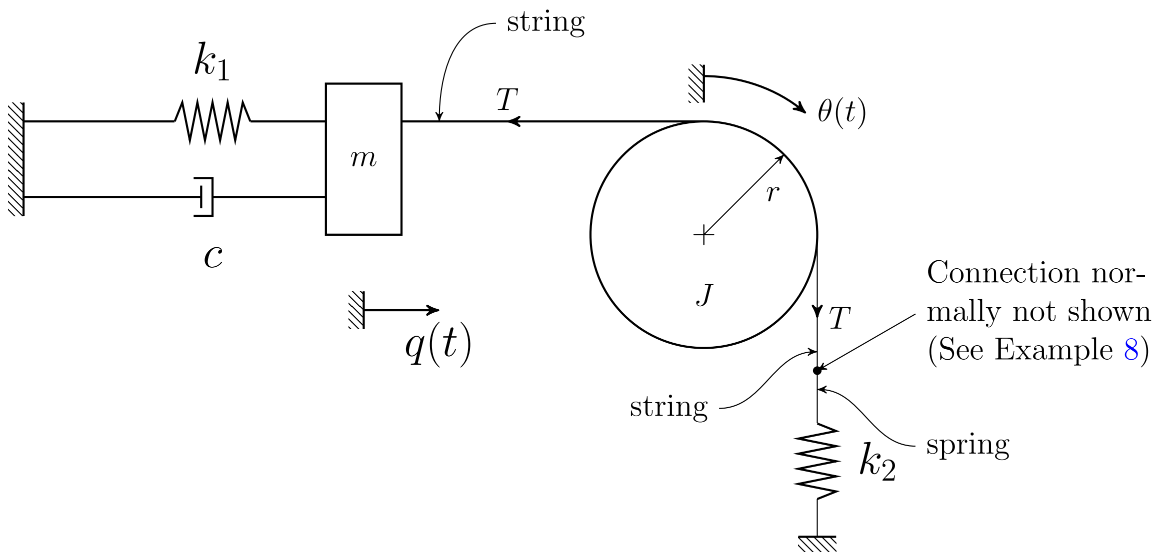


Figure 6: The mass-spring-damper system of Figure 5 connected to a spring on the right through a string wrapped around a pulley. The tension in the string is T, which is the same at all points for an ideal string. Assuming the spring is always in tension ($T \geq 0$), then any vertical motion of the upper end-point of spring k_2 causes an equal motion of the mass m, and *vice versa*. The force f_{s_2} on m due to spring k_2 is $f_{s_2} = -k_2 q$. Make sure you understand why this force is NOT $k_2 q$.

6 Examples

Example 2 (Spring-Mass-Damper)**.** Find the EOM for the System shown below.

□

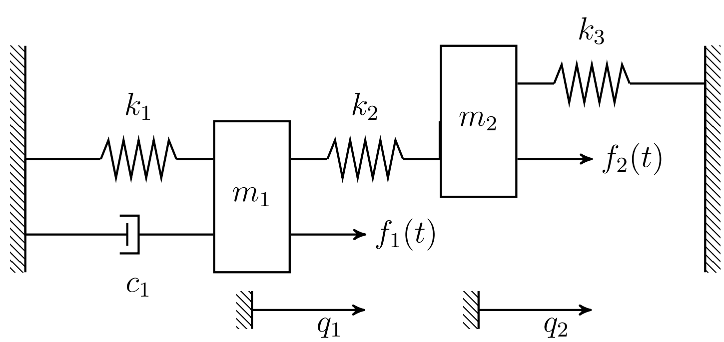
Solution: Since no information is provided, we assume spring and damper are linear, and that the spring has zero extension when $q = 0$.

Applying NSL: $m\ddot{q}(t) = f(t) - f_s - f_d = f(t) - kq(t) - c\dot{q}(t)$, so that we get

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = f(t).$$

We will see this kind of second-order system many times.

Example 3. Find the EOM for the System shown below.

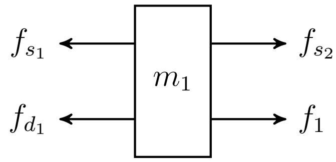


□

Solution: There are two independent masses, so that the system is described by two independent positions q_1 and q_2. We would therefore expect two differential equations to model this system.

Since no information is provided, we assume springs and dampers are linear, and that the springs have zero extension when $q_1 = 0$ and $q_2 = 0$.

Mass 1 The position of this mass is given by $q_1(t)$. The free body diagram is



Spring k_1 By our convention,

$$f_{s_1} = k_1 q_1(t).$$

Damper c_1 By a similar convention,

$$f_{d_1} = c_1 \dot{q}_1(t).$$

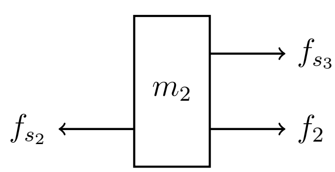
Spring k_2 The change in length of the spring from its free length is $q_2(t) - q_1(t)$, so that

$$f_{s_2} = k_2(q_2(t) - q_1(t)).$$

Newton's Second Law

$$\begin{aligned} m_1 \ddot{q}_1(t) &= \sum f \\ &= f_{s_2} + f_1 - f_{s_1} - f_{d_1} \\ &= k_2 q_2(t) - k_2 q_1(t) + f_1(t) - k_1 q_1(t) - c_1 \dot{q}_1(t) \end{aligned} \tag{12}$$

Mass 2 The position of this mass is given by $q_2(t)$. The free body diagram is



Spring k_3 By our convention,

$$f_{s_3} = -k_3 q_2(t).$$

Newton's Second Law

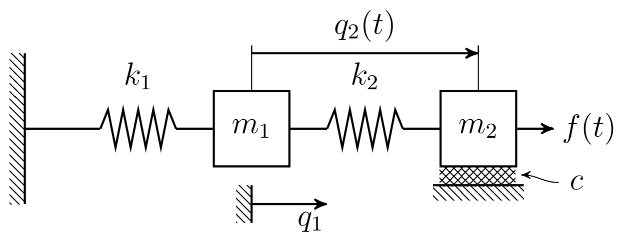
$$\begin{aligned} m_2\ddot{q}_2(t) &= \sum f \\ &= f_{s_3} + f_2 - f_{k_2} \\ &= -k_3 q_2(t) + f_2(t) - k_2 q_2(t) + k_2 q_1(t) \end{aligned} \tag{13}$$

Combined Equations of Motion (EoM)

$$m_1\ddot{q}_1(t) + c_1\dot{q}_1(t) + (k_2 + k_1)q_1(t) - k_2 q_2(t) = f_1(t) \tag{14}$$

$$m_2\ddot{q}_2(t) - k_2 q_1(t) + (k_2 + k_3)q_2(t) = f_2(t) \tag{15}$$

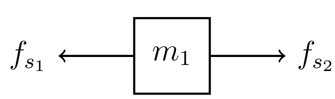
Example 4. Derive the equations of motion of for the system below. The spring is at its free length (extension is zero) when $q_2 = d$.



□

There are two independent masses, so that the system is described by two independent positions. We again expect two differential equations to model this system.

Mass 1 The position of mass m_1 is $q_1(t)$. The FBD is



Applying the usual conventions, we have

$$f_{s_1} = k_1 q_1(t).$$

The length of the spring k_2 is $q_2(t)$ and the free length is d. The extension or compression of the second spring is $q_2(t) - d$, so that under a linear model,

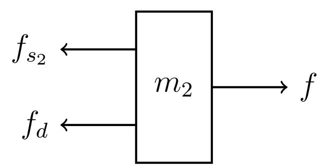
$$f_{s_2} = k_2(q_2(t) - d).$$

Applying Newton's second law, we get

$$
\begin{aligned}
m_1\ddot{q}_1(t) &= \sum f \\
&= f_{s_2} - f_{s_1} \\
&= k_2(q_2(t) - d) - k_1 q_1(t)
\end{aligned}
\tag{16}
$$

Mass 2 The position of mass m_2 is $\mathbf{q_1(t)} + \mathbf{q_2(t)}$, since the quantity $q_2(t)$ is defined with respect to a moving frame with translational position $q_1(t)$, and $q_1(t)$ is defined relative to an inertial frame.

The free body diagram is



The friction force is due to linear damping with coefficient c, and the velocity of m_2 is $\dot{q}_1(t) + \dot{q}_2(t)$. Therefore,

$$f_d = c\dot{q}_1(t) + c\dot{q}_2(t).$$

Applying Newton's second law, we get

$$
\begin{aligned}
m_2(\ddot{q}_1(t) + \ddot{q}_2(t)) &= \sum f \\
&= f - f_{s_2} - f_d \\
&= -k_2(q_2(t) - d) - c\dot{q}_1(t) - c\dot{q}_2(t)
\end{aligned}
\tag{17}
$$

Combined EoM

$$m_1\ddot{q}_1(t) + k_1 q_1(t) - k_2(q_2(t) - d) = 0 \tag{18}$$

$$m_2\ddot{q}_1(t) + m_2\ddot{q}_2(t) + c\dot{q}_1(t) + c\dot{q}_2(t) + k_2(q_2(t) - d) = f(t) \tag{19}$$

Example 5 (Suspension Model With Gravity)**.** Write down the equations of motion of the system: Find the equilibria, and rewrite the EoMs in terms of relative displacements. □

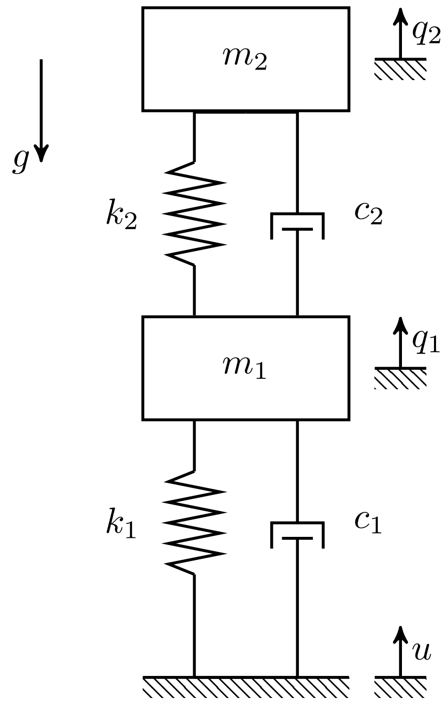
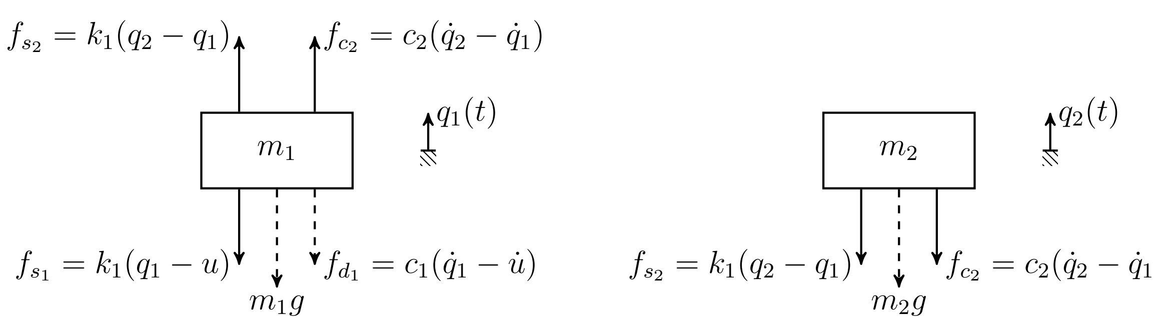


Figure 7: Suspension Model

Solution: We have two masses, and we therefore need two free body diagrams.

For mass m_1, we have five forces: two from two springs, two from two dampers, and gravity. For mass m_2, we have three forces: one from one spring k_2, one from one damper c_2, and gravity.



Spring k_1 **and Damper** c_1 Since these two elements are connected between m_1 and the reference with position $u(t)$, their extension is proportional to $q_1 - u$. If q_1 is positive, lengths of both elements increase. If u is positive, lengths of both elements decrease. Therefore, assuming linear spring and dampers, $f_{s_1} = k_1(q_1 - u)$, and $f_{d_1} = c_1(\dot{q}_1 - \dot{u})$.

Spring k_2 and Damper c_2 Since these two elements are connected between m_1 and m_2, the extension of these elements is $q_2 - q_1$. So, $f_{s_2} = k_2(q_2 - q_1)$, and $f_{d_2} = c_2(\dot{q}_2 - \dot{q}_1)$.

Applying Newton's Second Law to m_1:

$$\begin{aligned} m_1\ddot{q}_1 &= \sum f && (20)\\ &= f_{s_2} + f_{d_2} - f_{s_1} - f_{d_1} - m_1 g && (21)\\ &= k_2(q_2 - q_1) + c_2(\dot{q}_2 - \dot{q}_1) - k_1(q_1 - u) - c_1(\dot{q}_1 - \dot{u}) - m_1 g && (22) \end{aligned}$$

Applying Newton's Second Law to m_2:

$$\begin{aligned} m_2\ddot{q}_2 &= \sum f && (23)\\ &= -f_{s_2} - f_{d_2} - m_2 g && (24)\\ &= -k_2(q_2 - q_1) - c_2(\dot{q}_2 - \dot{q}_1) - m_2 g && (25) \end{aligned}$$

Combined equations of motion:

$$\begin{aligned} m_1\ddot{q}_1 &= k_2(q_2 - q_1) + c_2(\dot{q}_2 - \dot{q}_1) - k_1(q_1 - u) - c_1(\dot{q}_1 - \dot{u}) - m_1 g && (26)\\ m_2\ddot{q}_2 &= -k_2(q_2 - q_1) - c_2(\dot{q}_2 - \dot{q}_1) - m_2 g && (27) \end{aligned}$$

Unforced Equilibria. We have two variables q_1 and q_2. To find the equilibria, we set the inputs to zero:

$$\begin{aligned} m_1\ddot{q}_1 &= k_2(q_2 - q_1) + c_2(\dot{q}_2 - \dot{q}_1) - k_1(q_1 - 0) - c_1(\dot{q}_1 - 0) - m_1 g && (28)\\ m_2\ddot{q}_2 &= -k_2(q_2 - q_1) - c_2(\dot{q}_2 - \dot{q}_1) - m_2 g && (29)\\ &\text{or} \\ m_1\ddot{q}_1 &= k_2(q_2 - q_1) + c_2(\dot{q}_2 - \dot{q}_1) - k_1(q_1) - c_1(\dot{q}_1) - m_1 g && (30)\\ m_2\ddot{q}_2 &= -k_2(q_2 - q_1) - c_2(\dot{q}_2 - \dot{q}_1) - m_2 g && (31) \end{aligned}$$

We replace $q_1(t) \to q_{1e} \implies \dot{q}_1 \to 0, \ddot{q}_1 \to 0$ and
$q_2(t) \to q_{2e} \implies \dot{q}_2 \to 0, \ddot{q}_2 \to 0$:

$$\begin{aligned} m_1 0 &= k_2(q_{2e} - q_{1e}) + c_2(0 - 0) - k_1(q_{1e}) - c_1(0 - 0) - m_1 g && (32)\\ m_2 0 &= -k_2(q_{2e} - q_{1e}) - c_2(0 - 0) - m_2 g && (33)\\ &\text{or} \\ 0 &= k_2 q_{2e} - (k_1 + k_2) q_{1e} - m_1 g && (34)\\ 0 &= -k_2 q_{2e} + k_2 q_{1e} - m_2 g && (35) \end{aligned}$$

We must solve these equations above to obtain q_{1e} and q_{2e} as expressions of only the system parameters and g. Adding the two equations leads to:

$$0 = k_1 q_{1e} - (m_1 + m_2)g \tag{36}$$

$$\implies q_{1e} = \frac{(m_1 + m_2)g}{k_1} \tag{37}$$

$$\text{Now, } 0 = -k_2 q_{2e} + k_2 q_{1e} - m_2 g \implies q_{2e} = q_{1e} - \frac{m_2 g}{k_2} \tag{38}$$

$$= \frac{(m_1 + m_2)g}{k_1} - \frac{m_2 g}{k_2} \tag{39}$$

To derive the EoM, since $\delta q_i = q_i(t) - q_{ie}$, we substitute

$$q_1(t) \to \delta q_1(t) + q_{1e} \implies \dot{q}_1(t) \to \delta\dot{q}_1(t), \ddot{q}_1(t) \to \delta\ddot{q}_1(t) \tag{40}$$

$$q_2(t) \to \delta q_2(t) + q_{2e} \implies \dot{q}_2(t) \to \delta\dot{q}_2(t), \ddot{q}_2(t) \to \delta\ddot{q}_2(t) \tag{41}$$

This substitution leads to

$$m_1\ddot{q}_1 = k_2(q_2 - q_1) + c_2(\dot{q}_2 - \dot{q}_1) - k_1(q_1 - u) - c_1(\dot{q}_1 - \dot{u}) - m_1 g, \text{ and} \tag{42}$$

$$m_2\ddot{q}_2 = -k_2(q_2 - q_1) - c_2(\dot{q}_2 - \dot{q}_1) - m_2 g, \tag{43}$$

which in turn becomes

$$m_1\delta\ddot{q}_1 = k_2(\delta q_2 + q_{2e} - \delta q_1 - q_{1e}) + c_2(\delta\dot{q}_2 - \delta\dot{q}_1) - k_1(\delta q_1 + q_{1e} - u)\cdots \tag{44}$$

$$- c_1(\delta\dot{q}_1 - \dot{u}) - m_1 g, \text{ and} \tag{45}$$

$$m_2\delta\ddot{q}_2 = -k_2(\delta q_2 + q_{2e} - \delta q_1 - q_{1e}) - c_2(\delta\dot{q}_2 - \delta\dot{q}_1) - m_2 g. \tag{46}$$

Finally, we simplify this to

$$m_1\delta\ddot{q}_1 = k_2(\delta q_2 - \delta q_1) + c_2(\delta\dot{q}_2 - \delta\dot{q}_1) - k_1(\delta q_1 - u) - c_1(\delta\dot{q}_1 - \dot{u}) \tag{47}$$

$$+ k_2(q_{2e} - q_{1e}) - k_1 q_{1e} - m_1 g, \text{ and} \tag{48}$$

$$m_2\delta\ddot{q}_2 = -k_2(\delta q_2 - \delta q_1) - c_2(\delta\dot{q}_2 - \delta\dot{q}_1) - k_2(q_{2e} - q_{1e}) - m_2 g. \tag{49}$$

The parts in red are equal to zero, according to (34) and (35). The parts in blue look similar to the original EoM, but without terms involving gravity! Finally,

$$m_1\delta\ddot{q}_1 = k_2(\delta q_2 - \delta q_1) + c_2(\delta\dot{q}_2 - \delta\dot{q}_1) - k_1(\delta q_1 - u) - c_1(\delta\dot{q}_1 - \dot{u}) \tag{50}$$

$$m_2\delta\ddot{q}_2 = -k_2(\delta q_2 - \delta q_1) - c_2(\delta\dot{q}_2 - \delta\dot{q}_1) \tag{51}$$

Example 6 (Spring Connected To Damper)**.** Write down the equations of motion of the system: □

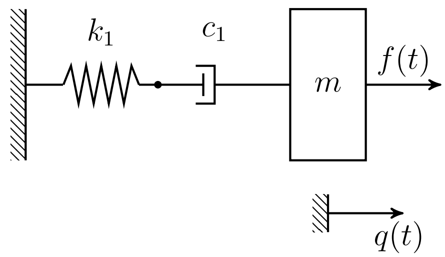


Figure 8: System with serially connected spring and dashpot.

Solution: Since there is only one mass, m, we need only one FBD. However, we cannot know the forces in spring k_1 and damper c_1 without knowing the extension in those elements. To figure out the extension, we name the point of connection between them as A, and assign a position q_1 to A, as shown:

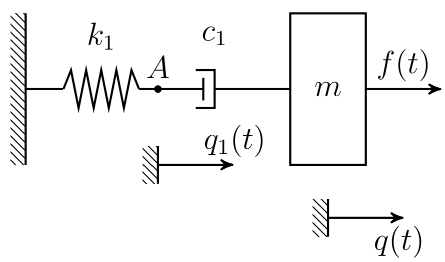
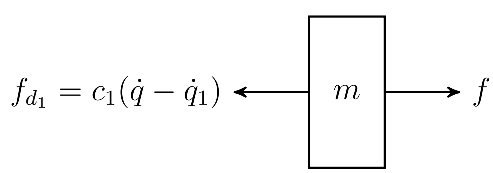


Figure 9: System with serially connected spring and dashpot. Add a position at A.

By creating q_1, we may write the extension of the spring k_1 as q_1, and the extension of the damper c_1 as $q - q_1$.

FBD of Mass m:



Applying Newton's Second Law to m:

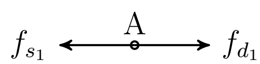
$$
\begin{aligned}
m\ddot{q} &= \sum f && (52)\\
&= f - f_{d_1} && (53)\\
&= f - c_1(\dot{q} - \dot{q}_1) && (54)
\end{aligned}
$$

An important question: is q_1 a part of the input to the system, or is it a part of the system's state?
Answer: It is not an input, since nothing external to the system can set $q_1(t)$ to a value, which is what characterizes an input.

Since it is not an input, how do we deal with it in state-variable equations or input-output differential equations?

The answer is that we relate it to q through a equation arising out of the behavior at A. If this equation is an algebraic relationship between q and q_1, then q_1 is not an independent state. If this equation is a differential equation that is different from the one due to NSL at m, then we have identified another state of the system.

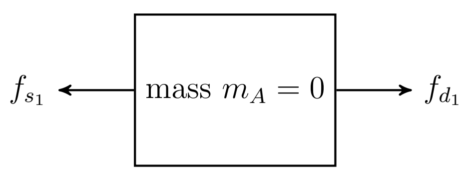
Ideal connectors are perfectly rigid, so they instantaneously transmit forces from one end to the other. This means that the forces on both 'ends' are equal. Applying this principle to the point A, we may draw:



Since the forces at the ends of this ideal connector (of zero width) are equal, we get,

$$f_{s_1} = f_{d_1}.$$

Equivalently, we can pretend that a mass m_A exists at A, but later use the fact that $m_A = 0$. FBD for m_A:



$$\text{NSL:} \quad m_a \times \ddot{q}_1 = \sum f \tag{55}$$
$$\implies 0 \times \ddot{q}_1 = f_{d_1} - f_{s_1} \tag{56}$$
$$\implies 0 = f_{d_1} - f_{s_1} \tag{57}$$
$$\implies f_{d_1} = f_{s_1} \tag{58}$$

Therefore, whichever way we choose to deal with what happens at A (treat it as perfectly rigid connector or massless connector), we get the same equation:

$$f_{d_1} = f_{s_1} \tag{59}$$
$$\implies c_1(\dot{q} - \dot{q}_1) = k_1 q_1 \tag{60}$$
$$\implies c_1 \dot{q} = k_1 q_1 + c_1 \dot{q}_1 \tag{61}$$

Therefore, we obtain an additional first-order ODE corresponding to q_1 being a state of the system.

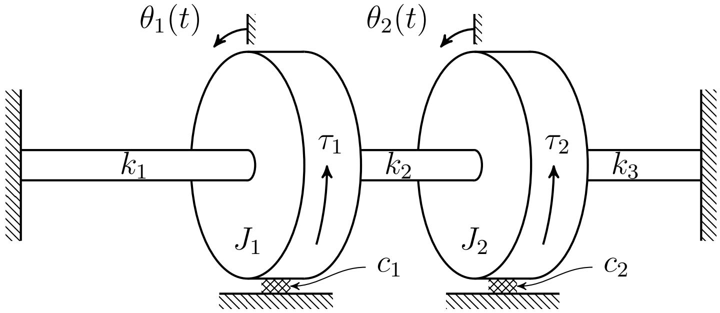
Collecting these differential equations, we get the EoM:

$$m\ddot{q} + c_1\dot{q} = f + c_1\dot{q}_1 \tag{62}$$

$$c_1\dot{q}_1 + k_1 q_1 = c_1\dot{q} \tag{63}$$

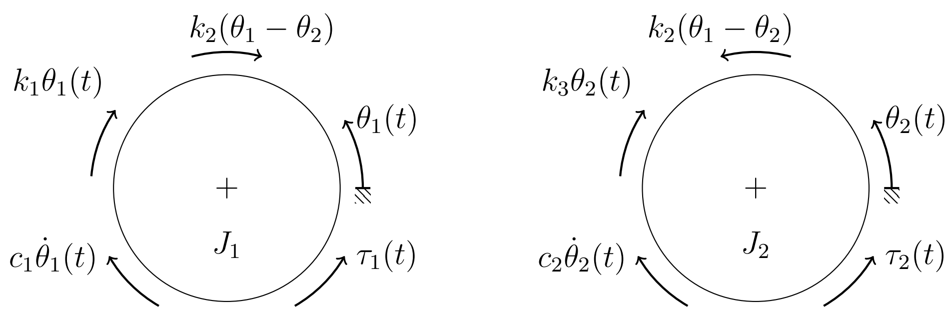
We may take the state x to be $x_1 = q$, $x_2 = \dot{q}$, and $x_3 = q_1$, input $u(t)$ is $f(t)$, and output y is $q_1 - q$.

Example 7 (Two DOF Rotational System)**.** Write down the equations of motion of the following rotating bodies:



Assume that $J_1 = J_2 = J$, $k_1 = k_2 = k$, $c_1 = c_2 = 0$, and $k_3 = 0$. Further, assume $\tau_1 = 0$. Write down the input-output equations with input $u = \tau_2$ and output $y = \theta_1 - \theta_2$.

Solution: We apply the rules and conventions described in earlier sections and obtain the following free body diagrams for the two rotational inertias:



□

Apply Newton's second law to get

$$J_1\ddot{\theta}_1 = \tau_1 - k_1\theta_1 - c_1\dot{\theta}_1 - k_2(\theta_1 - \theta_2) \tag{64}$$

$$J_2\ddot{\theta}_2 = \tau_2 - k_3\theta_2 - c_2\dot{\theta}_1 + k_2(\theta_1 - \theta_2) \tag{65}$$

We apply the substitutions given in the problem description to get

$$J\ddot{\theta}_1 = -k\theta_1 - k(\theta_1 - \theta_2) \tag{66}$$

$$J\ddot{\theta}_2 = \tau_2 + k(\theta_1 - \theta_2) \tag{67}$$

and then

$$J\ddot{\theta}_1 = -k\theta_1 - ky \tag{68}$$

$$J\ddot{\theta}_2 = u + ky \tag{69}$$

and finally, using the p-operator

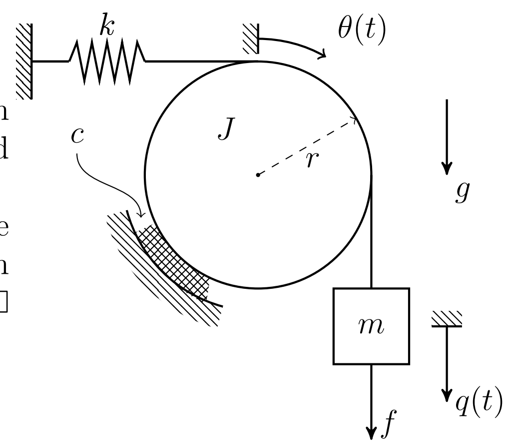
$$Jp^2\theta_1 = -k\theta_1 - ky \tag{70}$$
$$Jp^2\theta_2 = u + ky \tag{71}$$

How do we eliminate the terms θ_1 and θ_2? We use the equation for the output: $y = \theta_1 - \theta_2$. We'll use the equations above to substitue for θ_1 and θ_2

$$(Jp^2 + k)\theta_1 = -ky \implies \qquad \theta_1 = \frac{-ky}{(Jp^2 + k)} \tag{72}$$
$$Jp^2\theta_2 = u + ky \implies \qquad \theta_2 = \frac{u + ky}{Jp^2} \tag{73}$$

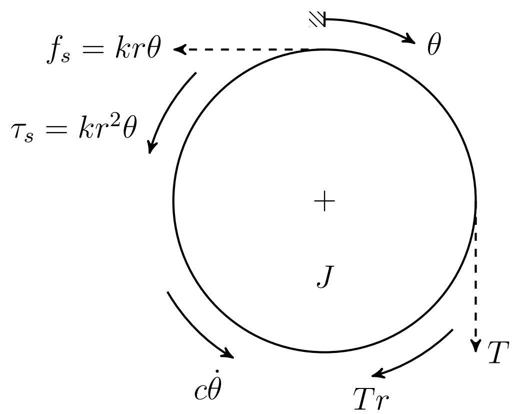
So, we get

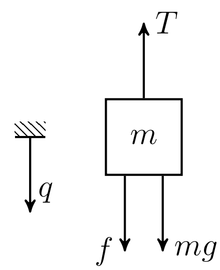
$$y = \theta_1 - \theta_2 \qquad \text{(by definition)} \tag{74}$$
$$\implies y = \frac{(-ky)}{(Jp^2 + k)} - \frac{(u + ky)}{Jp^2} \qquad \text{(substituting (72), (73))} \tag{75}$$
$$\implies (J^2p^4 + 3Jkp^2 + k^2)y = (Jp^2 + k)u \qquad \text{(algebraic manipulations)} \tag{76}$$
$$\implies J^2y^{(4)}(t) + 3Jk\ddot{y}(t) + k^2y(t) = J\ddot{u}(t) + ku(t) \qquad \text{(apply the } p\text{-operator)} \tag{77}$$



Example 8. Consider the mechanical system shown above. Assume that the cable around the cylinder neither stretches nor slips.
Find the EOM and express it in state-variable form, where the output is the force acting on the spring. □

Solution: Note that $q = r\theta$. The string tension creates torques that act on the rotational inertia J, and a force that acts on the mass m. Noting these points, we can write the the following free-body diagrams:





Applying Newton's second law,

$$J\ddot{\theta} = Tr - kr^2\theta - c\dot{\theta} \tag{78}$$

$$m\ddot{q} = f - T + mg \tag{79}$$

We define the state as $x_1 = \theta$, $x_2 = \dot{\theta}$.

Why do we exclude q**?** The answer is that q and θ are algebraically related, $q = r\theta$, so that they are not independent variables. Therefore, we must eliminate **both** q and T from (78) and (79).

$$J\ddot{\theta} = r\underbrace{(f - mr\ddot{\theta} + mg)}_{T \text{ (From (79))}} - kr^2\theta - c\dot{\theta} \tag{80}$$

$$\Longrightarrow (J + mr^2)\ddot{\theta} = fr + mgr - c\dot{\theta} - kr^2\theta \tag{81}$$

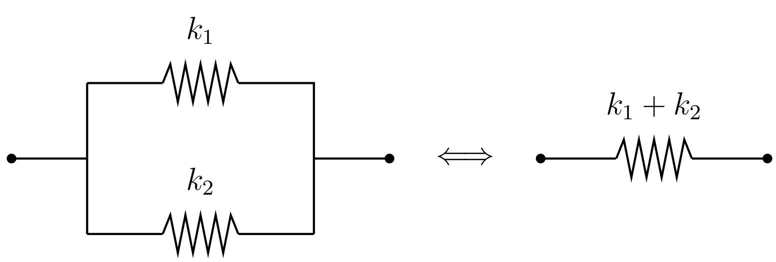
Now that we have the EoM in terms of θ, we may derive the state-variable equations. These would be

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \underline{\quad ? \quad} \\
y &= krx_1
\end{aligned}$$

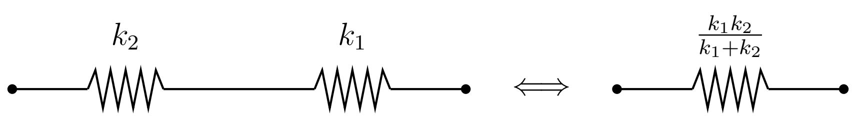
7 Series And Parallel

7.1 Linear Springs

Springs in Parallel:

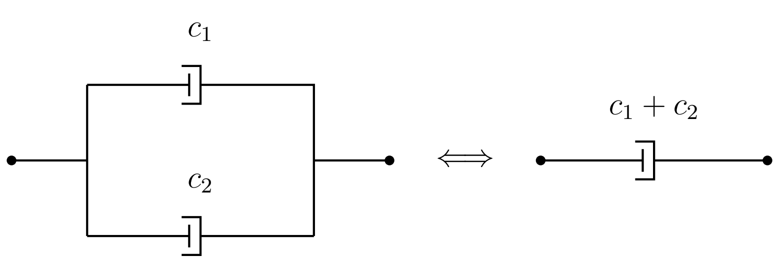


Springs in Series:

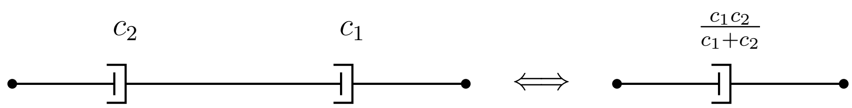


7.2 Linear Dampers

Dampers in Parallel:

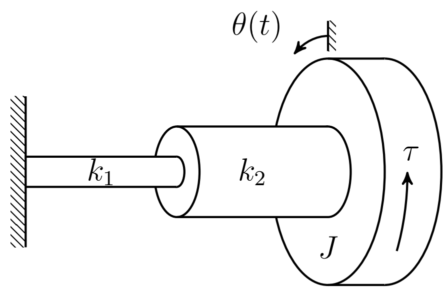


Dampers in Series:

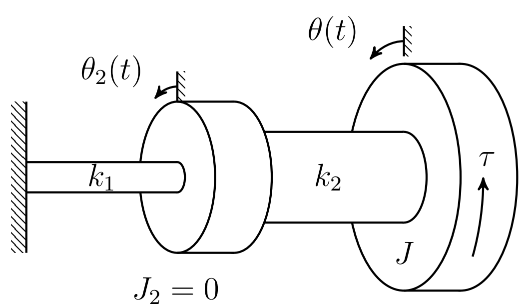


7.3 Torsional Springs

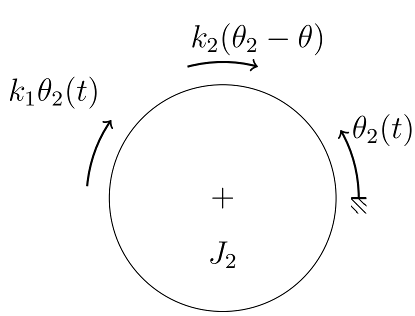
Torsional Springs in Series:



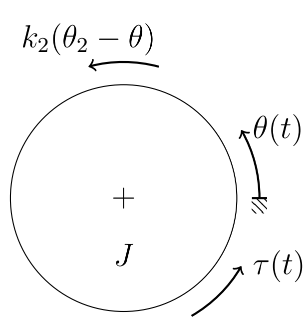
Let’s add a zero-inertia wheel between the two torsional spring elements.



Free-body Diagrams:



Free body Diagram J_2



Free body Diagram J

Applying Newton's laws, we get

$$J_2\ddot{\theta}_2 = -k_1\theta_2 - k_2(\theta_2 - \theta) \tag{82}$$

$$J\ddot{\theta} = \tau + k_2(\theta_2 - \theta) \tag{83}$$

Since $J_2 = 0$, we derive

$$k_2\theta = (k_1 + k_2)\theta_2 \qquad \text{(From (82))} \tag{84}$$

Therefore, we can substitute this expression into (83) to obtain

$$J\ddot{\theta} = \tau + k_2\theta_2 - k_2\theta \tag{85}$$

$$\implies J\ddot{\theta} = \tau + k_2\frac{k_2\theta}{k_1 + k_2} - k_2\theta \tag{86}$$

$$\implies J\ddot{\theta} = \tau + \frac{k_2^2 - k_1k_2 - k_2^2}{k_1 + k_2}\theta \tag{87}$$

$$\implies J\ddot{\theta} + \frac{k_1k_2}{k_1 + k_2}\theta = \tau \tag{88}$$

Comparing this expression to the single rotational mass-spring system in (11), we can see that the equivalent torsional stiffness is

$$k_{eq} = \frac{k_1k_2}{k_1 + k_2}.$$

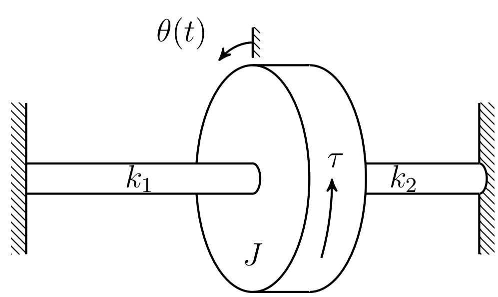
Torsional Springs in Parallel:

This situation is similar to that in Example 7. The FBD is straightforward, and leads to the dynamics

$$J\ddot{\theta} = \tau - k_1\theta - k_2\theta \tag{89}$$

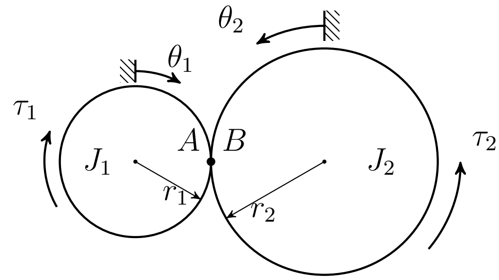
$$\implies J\ddot{\theta} + (k_1 + k_2)\theta = \tau \tag{90}$$

$$\implies k_{eq} = k_1 + k_2 \tag{91}$$



8 Gears

We assume that gears rotate without slipping of the teeth. The effect is that there exists an algebraic relationship between θ_1 and θ_2. There's a point on one gear (A) that's in contact with a point on the other gear (B) (an over-simplification of meshing of the teeth). The velocities of those two points must be exactly the same for no slippage to occur:



$$r_1\dot{\theta}_1 = r_2\dot{\theta}_2.$$

Furthermore, both gears experience an equal and opposite contact force. Therefore,

$$\dot{\theta}_1 = \left(\frac{r_2}{r_1}\right)\dot{\theta}_2 = N\dot{\theta}_2, \tag{92}$$

where N is the gear ratio. Integration of relationship (92) implies that

$$\theta_1 = N\theta_2 + \text{const.}$$

We often assume that the constant is zero, in which case

$$\theta_1 = N\theta_2.$$

Similarly, we may differentiate relationship (92) to arrive at

$$\ddot{\theta}_1 = N\ddot{\theta}_2.$$

It is also possible to define the gear ratio for the two gears as $N = r_1/r_2$, in which case $\dot{\theta}_1 = \dot{\theta}_2/N$. For each problem, we must **clearly specify the gear ratio** to avoid confusion.

8.1 Ideal Gears and Torque Ratios

We derive the equations of motion for the gear system as follows. An important term is the internal contact force along the common tangent to both gears. This contact force is equal and opposite, but applies differing torques on each gear.

Applying Newton's laws, we get

$$J_1\ddot{\theta}_1 = \tau_1 + Fr_1 \tag{93}$$

$$J_2\ddot{\theta}_2 = \tau_2 - Fr_2 \tag{94}$$

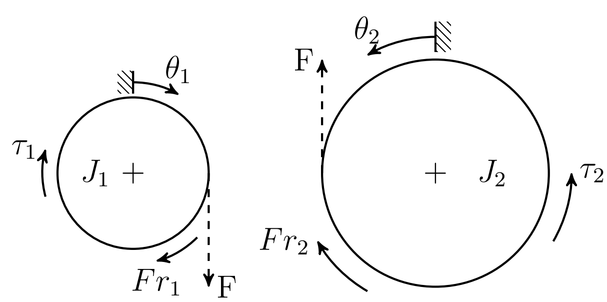


Figure 10: Free body diagrams

The gears do not slip, therefore

$$\ddot{\theta}_1 = N\ddot{\theta}_2,$$

where $N = r_2/r_1$.

Since θ_1 and θ_2 are not independent, we will be able to obtain the equations of motion in terms of only one of these angular positions by eliminating F from (93) and (94):

$$(N^2 J_1 + J_2)\ddot{\theta}_2(t) = \tau_2(t) + N\tau_1(t) \tag{95}$$

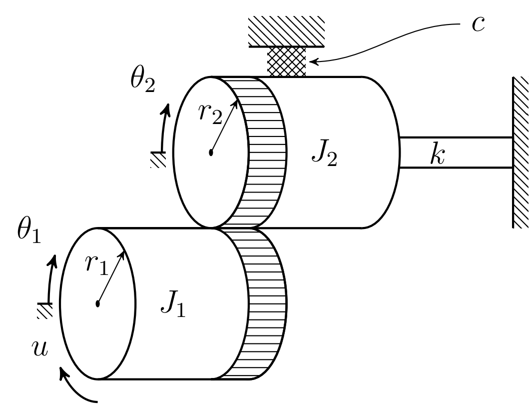
If these gears were ideal, meaning they are perfectly rigid without having any inertia, then we would have $J_1 = J_2 = 0$. When these two ideal gears are meshed, we get $\tau_2(t) = -N\tau_1(t)$. Therefore, **these ideal gears act as a torque amplifier** when $N > 1$ and τ_1 is a torque we generate and τ_2 is a torque that we want to apply on some other system. This amplification is independent of $\ddot{\theta}_2(t)$.

Suppose the two gears are rotating at constant velocity. Even if the gears are not ideal, meaning J_1, J_2 are non-zero, it turns out that $\tau_2 = -N\tau_1$, because $\ddot{\theta}_2 = 0$. Therefore, for two meshed gears that are either not rotating or rotating at constant velocity, the gears act as a torque amplifier when $N > 1$.

These considerations affect the design of robots, for example, where ideally we must drive the joints of the robot using low-inertia high-torque motors. To achieve a high torque output using lighter motors, we use gears to connect the motor output shaft to the link.

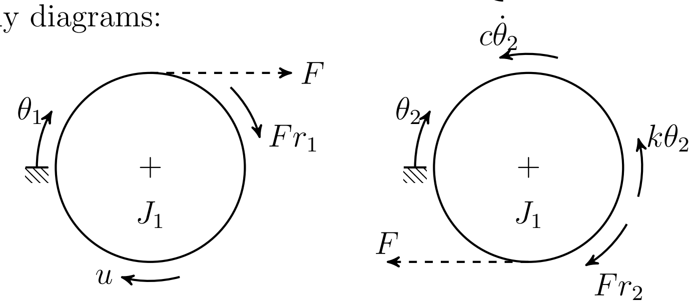
8.2 Example

Example 9. Consider the gears on the right, where we do not show the bearing within which gear J_1 rotates. The gears roll without slipping, so $r_1\dot{\theta}_1 = -r_2\dot{\theta}_2$. This negative sign accounts for the fact that if θ_1 increases, θ_2 must decrease for no slippage to occur. The gear ratio N is $N = \frac{\dot{\theta}_1}{\dot{\theta}_2} = -r_2/r_1$.



Write the input-output equations with input u and $y = \theta_1$. □

Solution: Free-body diagrams:



Note: In the FBD for J_1, we may choose F to point left instead of the right. This requires that in the FBD for J_2, we must have F point to the right. We still get the same final EoM once we eliminate F.

Applying Newton's laws to the two bodies, we get the Equations of Motion:

$$J_1\ddot{\theta}_1 = Fr_1 + u$$
$$J_2\ddot{\theta}_2 = -c\dot{\theta}_2 - k\theta_2 + Fr_2$$

substituting $\theta_1 \rightarrow y$, $\theta_2 \rightarrow \theta_1/N \rightarrow y/N$ (Make sure you understand why we use y/N instead of Ny) :

$$J_1\ddot{y} = Fr_1 + u \tag{96}$$
$$\frac{J_2}{N}\ddot{y} = -\frac{c}{N}\dot{y} - \frac{k}{N}y + Fr_2 \tag{97}$$

Now

$$\left(\frac{J_2}{N}\ddot{y} = -\frac{c}{N}\dot{y} - \frac{k}{N}y + Fr_2\right) \times \frac{1}{N} \rightarrow \frac{J_2}{N^2}\ddot{y} = -\frac{c}{N^2}\dot{y} - \frac{k}{N^2}y - Fr_1 \tag{98}$$

We just need to add Equation (96) to the one above to eliminate F:

$$\left(J_1 + \frac{J_2}{N^2}\right)\ddot{y} = -\frac{c}{N^2}\dot{y} - \frac{k}{N^2} + u \tag{99}$$

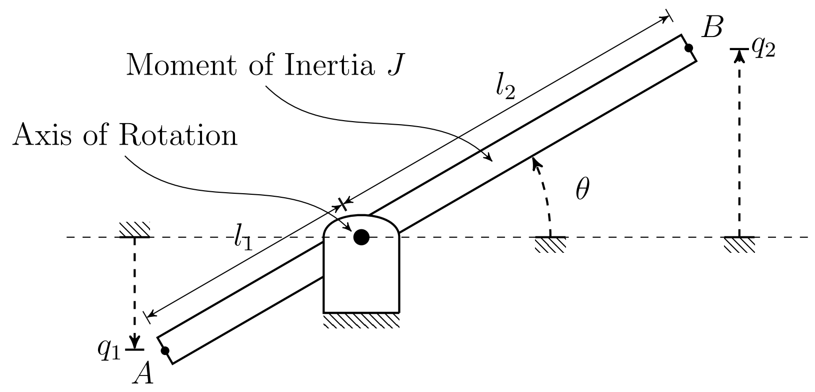
Finally,

$$(N^2J_1 + J_2)\ddot{y} + c\dot{y} + ky = N^2u \tag{100}$$

(**Check your understanding:** Compare the equation above to (95). How would you choose/modify output y and input u, and parameter values, to make them match?)

9 Levers

Consider the lever to the right. It consists of a rigid mass that has a length much greater than its other two dimensions, which rotates about an axis (fulcrum) perpendicular to its length.



We're interested in the vertical displacements of the end-points A and B, given by q_1 and q_2 respectively. These displacements depend on θ as

$$q_1 = l_1 \sin\theta, \quad q_2 = l_2 \sin\theta. \tag{101}$$

Therefore,

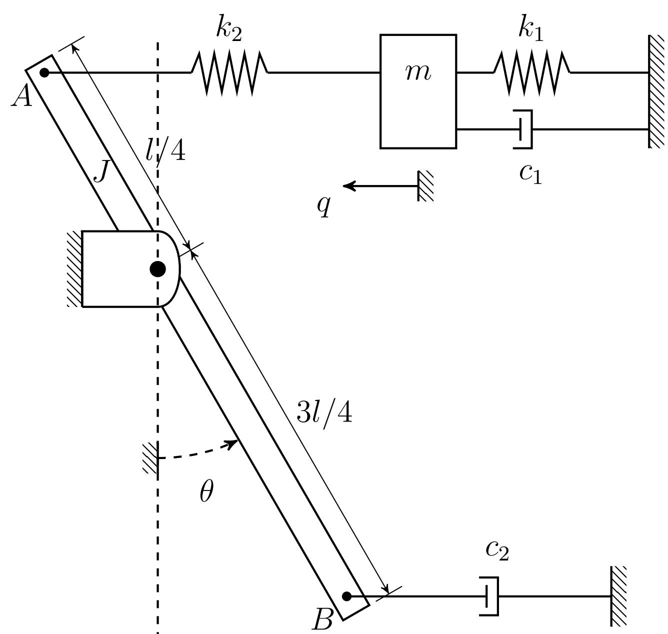
$$q_1 = \frac{l_1}{l_2} q_2, \text{ so that } \dot{q}_1 = \frac{l_1}{l_2}\dot{q}_2, \text{ and } \ddot{q}_1 = \frac{l_1}{l_2}\ddot{q}_2.$$

9.1 Small angle approximation

If $\theta \approx 0$, then $\sin\theta \approx 0$. Then $q_1 \approx l_1\theta$, and $q_2 \approx l_2\theta$. Importantly, the relationship is linear for small angles.

Example 10. Find the equations of motion for the lever and mass system on the right.

Use the small angle approximation to obtain equations linear in the positions. □



Solution: Since we will use the small-angle approximation, we treat the motion of the translating mass, spring, and dampers as being purely horizontal. More importantly, the points A and B are approximated as moving purely horizontally, despite being located on the purely rotating body J. This behavior is only valid under the small-angle approximation of motion.

Since no information is provided, we assume springs and dampers are linear, and that the springs have zero extension when $\theta = 0$ and $q = 0$. We know how to handle the forces on mass m due to spring k_1 and damper c_1.

Spring k_2. The spring extends when $\theta > 0$ and $q = 0$, or when $\theta = 0$ and $q < 0$. The extension due to rotation of J is $l\theta/4$ (small angle approximation). Therefore, the spring force in extension is

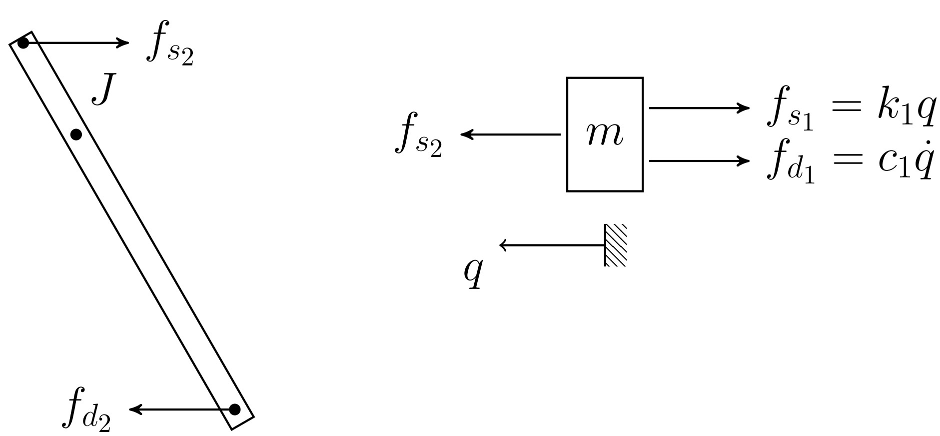
$$f_{s_2} = k_2\left(\frac{l\theta}{4} - q\right).$$

Damper c_2. The velocity of point b has a magnitude $3l\dot{\theta}/4$. When $\dot{\theta} > 0$, the inertia J is spinning counter-clockwise, and point B is moving to the right (assuming $\theta \approx 0$). We should then expect the damping force to be

$$f_{d_2} = c_2 3l\dot{\theta}/4 = \frac{3c_2 l}{4}\dot{\theta},$$

pointing to the *left* when indicated on the FBD of J.

Fee-body diagrams:



Apply Newton's second law:

$$J\ddot{\theta} = -f_{s_2}\frac{l}{4} - f_{d_2}\frac{3l}{4} = -k_2\left(\frac{l\theta}{4} - q\right)\left(\frac{l}{4}\right) - \left(\frac{3c_2 l}{4}\dot{\theta}\right)\left(\frac{3l}{4}\right) = -\frac{k_2 l^2}{16}\theta + \frac{k_2 l}{4}q - \frac{c_2 9l^2}{16}\dot{\theta}$$

$$m\ddot{q} = f_{s_2} - k_1 q - c_1\dot{q} = \frac{k_2 l}{4}\theta - k_2 q - k_1 q - c_1\dot{q}$$

Collecting terms where possible,

$$\begin{aligned}
J\ddot{\theta} + \frac{c_2 9l^2}{16}\dot{\theta} + \frac{k_2 l^2}{16}\theta &= \frac{k_2 l}{4} q \\
m\ddot{q} + c_1\dot{q} + (k_1 + k_2)q &= \frac{k_2 l}{4}\theta
\end{aligned}$$

It is important to remember that for this system q and θ are **not** algebraically related, since they are connected by a spring, not a rigid connection.