Laplace Transforms

Hasan A. Poonawala
Fall 2020

Contents

1 The Laplace Transform

1.1 Definition

Definition 1. Let $q(t)$ be a function of t, where $t \geq 0$. Then, the (one-sided) Laplace transform of $q(t)$ is given as

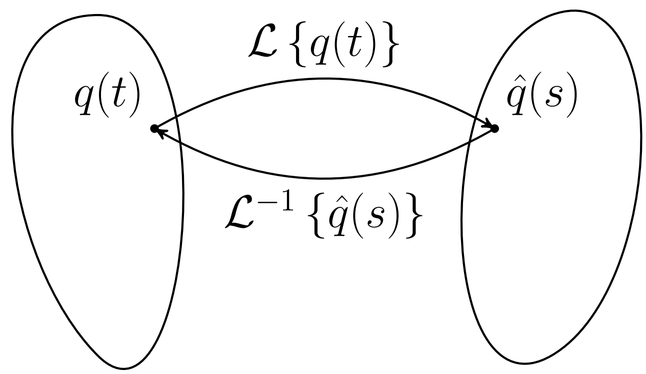
$$\hat{q}(s) = \mathcal{L}\left\{q(t)\right\} = \int_0^\infty e^{-st} q(t) dt.$$

The independent variable t, which is usually time, is a non-negative real variable.

The variable s is a complex variable, that is, $s \in \mathbb{C}$.

The Laplace transform is an **operator** which operates on functions in the time domain (dependent variable is time, a non-negative real number) and produces functions in the s-domain (a.k.a frequency domain), meaning they are functions of the complex variable s.

Under appropriate technical conditions, we can define the inverse $\mathcal{L}^{-1}\left\{\cdot\right\}$ of the Laplace transform of a function.



Space of time-domain functions

Space of s-domain functions

1.2 Examples of Laplace Transforms

Example 1 (Unit Step)**.** The unit step function is

$$H(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Taking the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{H(t)\} &= \int_0^\infty e^{-st}H(t)dt. \\
&= \int_0^\infty e^{-st}\ 1\ dt. \\
&= \left[-\frac{1}{s}e^{-st}\right]_{t=0}^{\infty} \\
&= 0 - \left(-\frac{1}{s}\right) \quad \text{(non-trivial limit as } t \to 0^+\text{)} \\
&= \frac{1}{s}
\end{aligned}$$

□

Example 2 (Exponential function)**.** The exponential function, parametrized by a is

$$q(t) = e^{-at}$$

Taking the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{q(t)\} &= \int_0^\infty e^{-st}e^{-at}dt. \\
&= \int_0^\infty e^{-(s+a)t}dt. \\
&= \left[-\frac{1}{s+a}e^{-(s+a)t}\right]_{t=0}^{\infty} \\
&= 0 - \left(-\frac{1}{s+a}\right) \quad \text{(non-trivial limit as } t \to 0^+\text{)} \\
&= \frac{1}{s+a}
\end{aligned}$$

□

Example 3 (Sinusoid function)**.** The exponential function, parametrized by a frequency ω rad/s is

$$q(t) = sin(\omega t).$$

Euler's formula states that for any real number a, $e^{ja} = \cos(a) + j\sin(a)$. Therefore,

$$q(t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

Table 1: Some important functions and their Laplace transforms

	$q(t)$	$\hat{q}(s) = \mathcal{L}\{q(t)\}$
unit impulse	$\delta(t)$	1
unit step	$H(t)$	$\frac{1}{s}$
	e^{-at}	$\frac{1}{s+1}$
	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
	t	$\frac{1}{s^2}$
	t^n	$\frac{n!}{s^{n+1}}$

Taking the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{q(t)\} &= \int_0^\infty e^{-st} sin(\omega t) dt. \\
&= \int_0^\infty e^{-st} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) dt. \\
&= \frac{1}{2j} \int_0^\infty e^{-st} e^{j\omega t} dt - \frac{1}{2j} \int_0^\infty e^{-st} e^{-j\omega t} dt. \\
&= \frac{1}{2j} \mathcal{L}\left\{ e^{-j\omega t} \right\} - \frac{1}{2j} \mathcal{L}\left\{ e^{j\omega t} \right\}
\end{aligned}$$

We know how to evaluate Laplace transforms of exponential functions:

$$\begin{aligned}
\mathcal{L}\{q(t)\} &= \frac{1}{2j} \frac{1}{(s - j\omega)} - \frac{1}{2j} \frac{1}{(s + j\omega)} \\
&= \frac{1}{2j} \left(\frac{s + j\omega - (s - j\omega)}{(s - j\omega)(s + j\omega)} \right) \\
&= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

□

Example 4 (Impulse function)**.** The impulse function, parametrized by time t_0 is

$$\delta(t - t_0) = \begin{cases} \infty & \text{if } t = t_0, \\ 0 & \text{if } t \neq t_0. \end{cases}$$

When $t_0 = 0$,

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0, \\ 0 & \text{if } t \neq 0. \end{cases}$$

Taking the Laplace transform:

$$\begin{aligned} \mathcal{L}\{q(t)\} &= \int_0^\infty e^{-st}\delta(t)dt. \\ &= e^{-st}\big|_{t=0}. \\ &= 1 \end{aligned}$$

□

1.3 But What Is It?

This section describes one way to interpret a Laplace transform.

Imagine you wanted to describe $q(t)$ to someone. One way to do so is to provide the value of $q(t)$ at every t, of which there are uncountably many.

Instead, note that a signal $q(t)$ may be approximated by a Maclaurin series (Taylor series expanded at 0):

$$q(t) = \sum_{n=0}^{\infty} \frac{q^{(n)}(0)}{n!} t^n, \tag{1}$$

where $q^{(n)}(t)$ is the n^{th} derivative of $q(t)$ with respect to t, and $n!$ is $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$. The function $q(t)$ is approximated by a polynomial in t, with coefficients that depend on the derivatives of $q(t)$ at 0. We now use a countably infinite set of numbers, instead of an uncountably infinite set, to describe $q(t)$. For example,

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots.$$

To communicate e^t, we would send $1, 1/2, 1/6, \ldots$.

That's still a lot of numbers. Is there a simpler way to communicate $q(t)$? One way is to hope that a simple rule generates the sequence of partial derivatives $q^{(n)}(0)$. For example, for the exponential above, $q^{(n)}(0) = 1$ for all $n \geq 0$!

The rule that generates the sequence $q^{(n)}(0)$ is related to **generating functions**. For many useful functions $q(t)$, the generating function in question is a rational function of s. We only need to communicate a finite set of coefficients to describe this rational function, and therefore $q(t)$, even for functions like e^t. The generating function is $\frac{1}{s}\hat{q}(\frac{1}{s})$, where $\hat{q}(s) = \mathcal{L}\{q(t)\}$.

Example 5 **($q(t) = e^{at}$).** Consider $\hat{q}(s) = \mathcal{L}\{e^{at}\} = \frac{1}{s+a}$.
The generating function is therefore $\frac{1}{s}\hat{q}(1/s) = \frac{1}{1-sa}$. This function corresponds to the series $\sum_{n\geq 0} a^n(s^n)$ so that $f^{(n)}(0) = a^n$. So, let's add up this series.

$$f(t) = a + at + \frac{a}{2}t^2 + \frac{a^3}{6}t^3 + \cdots \tag{2}$$

$$= 1 + at + \frac{a}{2}t^2 + \frac{a^3}{6}t^3 + \cdots \tag{3}$$

$$= e^{at}, \text{the intended function.} \tag{4}$$

□

Example 6 **($q(t) = \sin \omega t$).** $\hat{q}(s) = \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$. Therefore, the generating function will be

$$\frac{1}{s}\frac{\omega}{s^{-2}+\omega^2} = \frac{s\omega}{1+(s\omega)^2}.$$

To compute the series with coefficients $\{a_n\}$ that this function generates, we use the equation

$$\frac{s\omega}{1+(s\omega)^2} = \sum_{n=0}^{\infty} a_n s^n \tag{5}$$

$$\implies s\omega = (1+(s\omega)^2)\sum_{n=0}^{\infty} a_n s^n \tag{6}$$

$$\implies s\omega = \sum_{n=0}^{\infty} a_n s^n + (s\omega)^2 \sum_{n=0}^{\infty} a_n s^n \tag{7}$$

$$\implies \omega s = \sum_{n=0}^{\infty} a_n s^n + \omega^2 \sum_{n=0}^{\infty} a_n s^{n+2} \tag{8}$$

$$\implies \omega s = \underbrace{a_0 + a_1 s + \sum_{n=2}^{\infty} a_n s^n}_{\text{take out first two terms}} + \omega^2 \sum_{n=0}^{\infty} a_n s^{n+2} \tag{9}$$

$$\implies \omega s = \underbrace{a_0 + a_1 s + \sum_{n=2}^{\infty} a_n s^n}_{\text{separate out first two terms}} + \omega^2 \sum_{n=0}^{\infty} a_{n+2} s^{n+2} \tag{10}$$

$$\implies \omega s = a_0 + a_1 s + \underbrace{\sum_{n=0}^{\infty} a_{n+2} s^{n+2}}_{\text{can consistently rewrite numbering}} + \omega^2 \sum_{n=0}^{\infty} a_n s^{n+2} \tag{11}$$

$$\implies 0 = a_0 + (a_1 - \omega)s + \underbrace{\sum_{n=0}^{\infty}(a_{n+2} + \omega^2 a_n)s^{n+2}}_{\text{renumbering allows combination}} \tag{12}$$

We now invoke the following principle: if the equation above holds for multiple values of s (which it does), the coefficients of s^n must be zero. So,

$$a_0 = 0 \qquad \Longrightarrow a_0 = 0 \tag{13}$$
$$a_1 - \omega = 0 \qquad \Longrightarrow a_1 = \omega \tag{14}$$
$$a_{n+2} + \omega^2 a_n = 0 \text{ for } n \geq 2 \qquad \Longrightarrow a_{n+2} = -\omega^2 a_n \text{ for } n \geq 2 \tag{15}$$

The second equation is a recurrence relation. For example, $a_2 = -\omega^2 a_0 = 0$, and $a_3 = -\omega^2 a_1 = -\omega^3$, and so on. Therefore, we can write the function dictated by this series. Remember, this series corresponds to $q^{(n)}(0)$.

$$q(t) = q(0) + q^{(1)}(0)t + \frac{q^{(2)}(0)}{2!}t^2 + \frac{q^{(3)}(0)}{3!}t^3 + \frac{q^{(4)}(0)}{4!}t^4 + \frac{q^{(5)}(0)}{5!}t^5 + \cdots \tag{16}$$
$$= a_0 + a_1 t + \frac{a_2}{2!}t^2 + \frac{a_3}{3!}t^3 + \frac{a_4}{4!}t^4 + \frac{a_5}{5!}t^5 + \cdots \tag{17}$$
$$= 0 + \omega t + 0 \cdot t^2 + \frac{-\omega^3}{3!}t^3 + 0 \cdot t^4 + \frac{\omega^5}{5!}t^5 + \cdots \tag{18}$$
$$= \omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \frac{(\omega t)^7}{7!} + \cdots \tag{19}$$
$$= \sin(\omega t), \quad \text{the intended function.} \tag{20}$$

□

To summarize, we may think of the Laplace Transform of a time-domain function as a compact representation of that function. In particular, it provides a way to construct all the terms in the Maclaurin series that corresponds to $q(t)$. The next section provides additional properties that make this complex-domain representation useful for manipulating functions of time.

2 Properties of the Laplace Transform

We look at the way the Laplace transform behaves when we modify its argument through:

- linear combinations
- differentiation
- multiplication by exponential e^{-at}
- multiplication by time t
- introducing a time delay
- integration

These behaviors allow us to evaluate Laplace transforms of arbitrary functions of time using Laplace transforms of simple functions.

2.1 Linear Combinations

The Laplace transform is a linear operator,

$$\mathcal{L}\{\alpha_1 q_1(t) + \alpha_2 q_2(t)\} = \alpha_1 \mathcal{L}\{q_1(t)\} + \alpha_2 L\{q_2(t)\}.$$

If we know that $\hat{q}_1(s) = \mathcal{L}\{q_1(t)\}$ and $\hat{q}_2(s) = \mathcal{L}\{q_2(t)\}$, then we can immediately compute $\hat{q}_3(s) = \mathcal{L}\{q_3(t)\}$ when $q_3(t) = \alpha_1 q_1(t) + \alpha_2 q_2(t)$ without computing another Laplace transform, and instead computing

$$\hat{q}_3(s) = \alpha_1 \hat{q}_1(s) + \alpha_2 \hat{q}_2(s).$$

2.2 Differentiation

Let $\hat{q}(s) = \mathcal{L}\{q(t)\}$. Then

$$\mathcal{L}\{\dot{q}(t)\} = s\mathcal{L}\{q(t)\} - q(0) = s\hat{q}(s) - q(0).$$

One way to derive this expression is using integration by parts

$$\int_{\tau=a}^{\tau=b} u(\tau)\frac{dv(\tau)}{d\tau}d\tau = [u(\tau)v(\tau)]_{\tau=a}^{\tau=b} - \int_{\tau=a}^{\tau=b} \frac{du(\tau)}{d\tau}v(\tau)d\tau$$

applied to the definition of a Laplace transform:

$$\begin{aligned}
\mathcal{L}\{\dot{q}(t)\} &= \int_0^\infty e^{-st}\frac{dq(t)}{dt}dt \\
&= \left[e^{-st}q(t)\right]_{t=0}^{\infty} - \int_0^\infty (-se^{-st})q(t)dt \\
&= (0 - q(0)) + s\int_0^\infty e^{-st}q(t)dt \\
&= -q(0) + s\mathcal{L}\{q(t)\} \\
&= s\hat{q}(s) - q(0)
\end{aligned}$$

For higher order derivatives,

$$\mathcal{L}\{q^{(n)}(t)\} = s^n\hat{q}(s) - s^{n-1}q(0) - s^{n-2}\dot{q}(0) - s^{n-3}\ddot{q}(0) - \cdots - sq^{(n-2)}(0) - q^{(n-1)}(0)$$

2.3 s-shift

$$\mathcal{L}\{e^{-at}q(t)\} = \hat{q}(s+a)$$

For example,

$$\mathcal{L}\left\{\cos(\omega t)\right\} = \frac{s}{s^2+\omega^2},$$

so that

$$\mathcal{L}\left\{e^{-at}\cos(\omega t)\right\} = \frac{s+a}{(s+a)^2+\omega^2}.$$

2.4 Multiplication by time

If $\mathcal{L}\{q(t)\} = \hat{q}(s)$, then

$$\mathcal{L}\{tq(t)\} = -\frac{d}{ds}\hat{q}(s).$$

For example,

$$\begin{aligned}
\mathcal{L}\{te^{-at}\} &= -\frac{d}{ds}\left(\mathcal{L}\{e^{-at}\}\right) \\
&= -\frac{d}{ds}\left(\frac{1}{s+a}\right) \\
&= \frac{1}{(s+a)^2}
\end{aligned}$$

We could arrive at the same result by using the s-shift property:

$$\begin{aligned}
\mathcal{L}\{t\} &= \frac{1}{s^2} \\
\implies \mathcal{L}\{te^{-at}\} &= \frac{1}{(s+a)^2}
\end{aligned}$$

2.5 Time delay

If $\mathcal{L}\{q(t)\} = \hat{q}(s)$, then the Laplace transform of $q(t-\tau)$, which is $q(t)$ delayed by τ seconds, is

$$\mathcal{L}\{q(t-\tau)\} = e^{-s\tau}\hat{q}(s).$$

2.6 Integration

If $\mathcal{L}\{q(t)\} = \hat{q}(s)$, then

$$\mathcal{L}\left\{\int_0^t q(s)ds\right\} = \frac{1}{s}\hat{q}(s).$$

For example,

$$\mathcal{L}\left\{\int_0^t \cos(\omega\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{\cos(\omega t)\} = \frac{1}{s}\frac{s}{s^2+\omega^2} = \frac{1}{s^2+\omega^2}.$$

Check: $\int_0^t \cos(\omega\tau)d\tau = \frac{1}{\omega}\sin(\omega t)$, so that

$$\mathcal{L}\left\{\int_0^t \cos(\omega\tau)d\tau\right\} = \frac{1}{\omega}\mathcal{L}\{\sin(\omega t)\} = \frac{1}{\omega}\frac{\omega}{s^2+\omega^2} = \frac{1}{s^2+\omega^2}.$$

3 Solving Linear ODEs

The value of Laplace transforms also shows up when trying to solve linear ordinary differential equations. Suppose we have the differential equation

$$\dot{y}(t) + ay(t) = u(t), \tag{21}$$

where $y(t_0) = y_0$ and a is a real number.

Our goal is to obtain a solution $y(t)$ of (21) defined on some time interval $[t_0, t_{final}]$.

3.1 Homogenous and Particular Solutions

One approach is to search for a homogenous solution $y_H(t)$ and then a particular solution $y_P(t)$ (see supplementary slides), so that the solution $y(t)$ is

$$y(t) = y_H(t) + y_P(t).$$

The homogenous solution $y_H(t)$ is the solution to the linear ODE obtained by setting $u(t) \equiv 0$ in (21). We then form a polynomial known as the characteristic equations whose roots dictate what $y_H(t)$ is.

Then, we use the form of $y_H(t)$ and $u(t)$ to predict $y_P(t)$, and try and find a solution using the method of undetermined coefficients.

3.2 Direct Solution Using Convolution

Consider the derivative of the expression $e^{at}y(t)$ expression:

$$\begin{aligned}\frac{d}{dt}\left(e^{at}y(t)\right) &= \left(\frac{d}{dt}e^{at}\right)y(t) + e^{at}\left(\frac{d}{dt}y(t)\right) \\ &= ae^{at}y(t) + e^{at}\dot{y}(t) \\ &= e^{at}\dot{y}(t) + ae^{at}y(t) \\ &= e^{at}(\dot{y}(t) + ay(t))\end{aligned}$$

Now, consider solving the equation

$$\begin{aligned} & \frac{d}{dt}\left(e^{at}y(t)\right) = e^{at}u(t) \\ \implies \quad & e^{at}(\dot{y}(t) + ay(t)) = e^{at}u(t) \\ \implies \quad & \dot{y}(t) + ay(t) = u(t),\end{aligned}$$

because $e^{at} \neq 0$ for any t.

So, we see that to solve (21), we need to solve

$$\begin{aligned} & \frac{d}{dt}\left(e^{at}y(t)\right) = e^{at}u(t) \\ \implies \quad & e^{at}y(t) = y(0) + \int_0^t e^{a\tau}u(\tau)d\tau \\ \implies \quad & y(t) = e^{-at}y(0) + \int_0^t e^{a(\tau-t)}u(\tau)d\tau \\ & \quad = y_H(t) + y_P(t)!\end{aligned}$$

Problem: Computing $z(t) = \int_0^t e^{a(\tau-t)}u(\tau)d\tau$.

This computation is of the form $\int_0^t g(t-\tau)h(\tau)d\tau$, which is known as the convolution of two functions $g(t)$ and $h(t)$, that is, $\bar{z}(t) = (g * h)(t)$. This convolution is usually tedious and difficult to carry out.

Solution: One advantage of Laplace transforms is that the convolution of two functions of time is 'identical' to the algebraic product of their two Laplace transforms!

3.3 Laplace Transforms For Solving ODEs

To compute $z(t) = \int_0^t g(t-\tau)h(\tau)d\tau$, we solve the following expression:

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{g(t)\}\mathcal{L}\{h(t)\}\right\}.$$

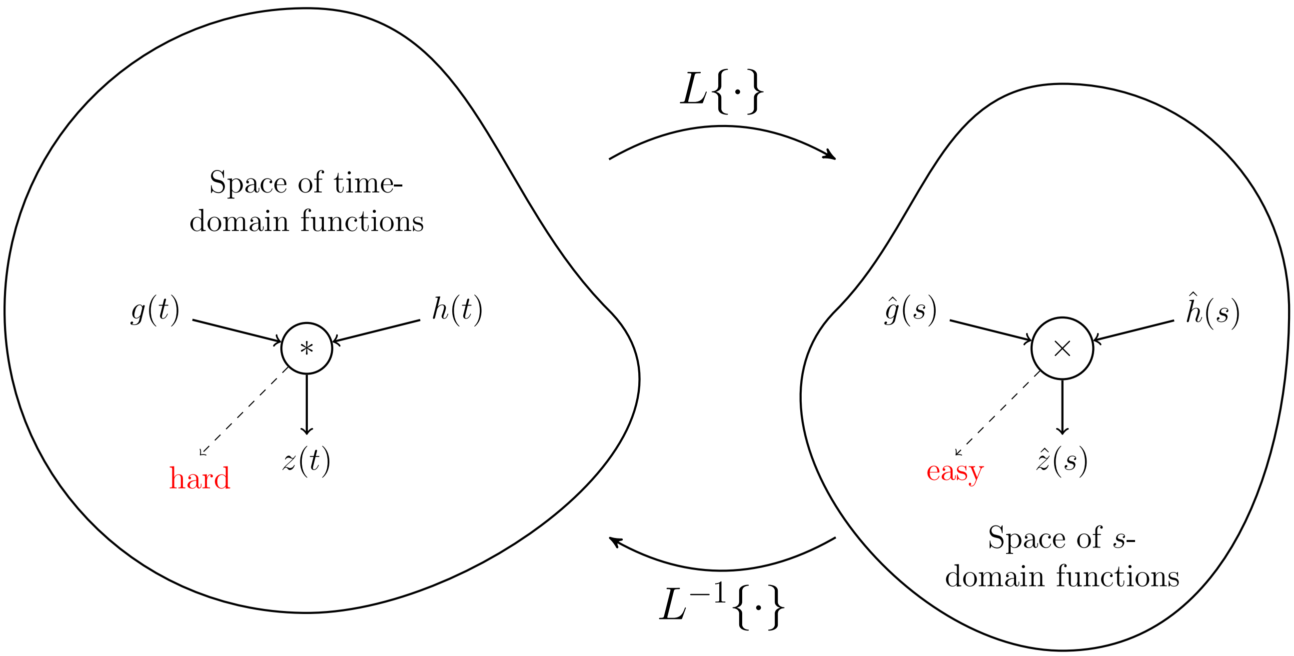


Figure 1: It is easier to implement the convolution operation involving two time-domain functions by computing the algebraic multiplication of their Laplace transforms, and then taking the inverse of the result.

In words, we convert the time-domain functions to s-domain functions, multiply these two s-domain functions, and then convert the result back into the time-domain. Figure 1 depicts this process.

Example 7 (Solving (21))**.** We will solve (21) for the case where $u(t) \equiv 1$ and $y(0) = y_0 = 0$. A direct solution is

$$
\begin{aligned}
y(t) &= e^{-at}y(0) + \int_0^t e^{a(\tau-t)}u(\tau)d\tau && (22)\\
&= 0 + \int_0^t e^{a(\tau-t)}1d\tau && (23)\\
&= e^{-at}\int_0^t e^{a\tau}d\tau && (24)\\
&= e^{-at}\left[\frac{1}{a}e^{at}\right]_{t=0}^{\infty} && (25)\\
&= \frac{1}{a} - \frac{e^{-at}}{a} && (26)
\end{aligned}
$$

To use Laplace transforms, first transform the ODE:

$$\begin{aligned}
&\dot{y}(t) + ay(t) = u(t) \\
\implies &\mathcal{L}\{\dot{y}(t) + ay(t)\} = \mathcal{L}\{u(t)\} \\
\implies &\mathcal{L}\{\dot{y}(t)\} + L\{ay(t)\} = \mathcal{L}\{u(t)\} \\
\implies &\mathcal{L}\{\dot{y}(t)\} + \mathcal{L}\{ay(t)\} = \mathcal{L}\{u(t)\} \\
\implies &s\hat{y}(s) - y(0) + a\hat{y}(s) = \hat{u}(s) \\
\implies &\hat{y}(s) = \frac{1}{s+a}y_0 + \frac{1}{s+a}\hat{u}(s),
\end{aligned}$$

where $\hat{y}(s) = \mathcal{L}\{y(t)\}$ and $\hat{u}(s) = \mathcal{L}\{u(t)\}$. We have that $y_0 = 0$, and $u(t) \equiv 1 \implies \hat{u}(s) = \frac{1}{s}$. Therefore,

$$\begin{aligned}
\hat{y}(s) &= \frac{1}{s+a} \times \frac{1}{s} \\
&= \frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)
\end{aligned}$$

Taking the Laplace inverse of both sides,

$$\begin{aligned}
\mathcal{L}^{-1}\{\hat{y}(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)\right\} \\
&= \frac{1}{a}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{a}L^{-1}\left\{\frac{1}{s+a}\right\} \\
&= \frac{1}{a} - \frac{e^{-at}}{a}
\end{aligned}$$

□

At present, the Laplace transform method seems longer. Let's change the control to $u(t) = t \implies \hat{u}(s) = 1/s^2$. Then,

$$\hat{y}(s) = \frac{1}{s^2(s+a)} = \frac{1}{a^2}\left(\frac{a}{s^2} - \frac{1}{s} + \frac{1}{s+a}\right)$$

so that we get

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{a^2}\left(\frac{a}{s^2} - \frac{1}{s} + \frac{1}{s+a}\right)\right\} = \frac{1}{a^2}\left(at - 1 + e^{-at}\right).$$

A direct solution is

$$\begin{aligned}
y(t) &= e^{-at}y(0) + \int_0^t e^{a(\tau - t)}u(\tau)d\tau \\
&= e^{-at}y(0) + \int_0^t \tau e^{a(\tau - t)}d\tau
\end{aligned}$$

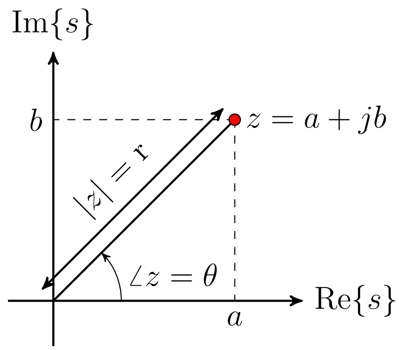
4 Review of Complex Numbers

Let $j^2 = -1$, or equivalently, $j = \sqrt{-1}$.

We represent a complex number $z \in \mathbb{C}$ in two ways.

The first is $z = a + jb$, where a and b are real numbers. We refer to a and b as the real and imaginary part of z respectively. We denote these parts of z as $\text{Re}\{z\}$ $(= a)$ and $\text{Im}\{z\}$ $(= b)$.

The second is $z = re^{j\theta}$, where r and θ are real numbers. The numbers r and θ are the magnitude and argument of z respectively.



Note that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, so that

$$\text{Re}\{z\} = a = r\cos\theta, \quad \text{Im}\{z\} = b = r\sin\theta.$$

To any complex number $z = a + jb = re^{j\theta}$, we can associate the following quantities:

- a magnitude $|z| = \sqrt{a^2 + b^2} = r$,
- an argument

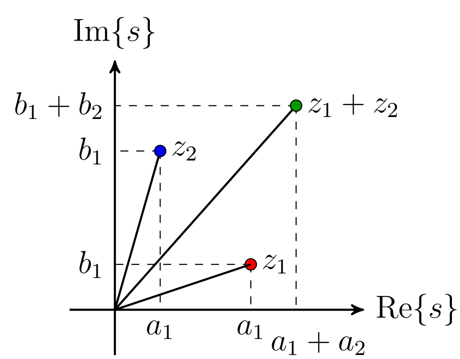
$$\angle z = \theta = \begin{cases} \tan^{-1}\frac{b}{a} & \text{if } a > 0 \\ \pi + \tan^{-1}\frac{b}{a} & \text{if } a < 0 \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0 \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0 \end{cases}.$$

- a complex conjugate $\bar{z} = a - jb$, and

Just as for real numbers, we can define the operations of addition and multiplication, which depend on the same operations that are defined for real numbers.

Addition. For two numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we define the sum

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2).$$

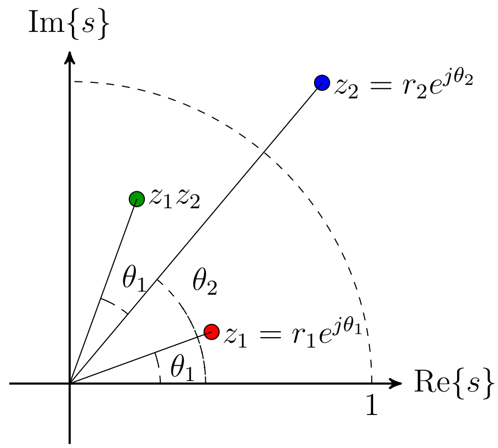


Multiplication. For two numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we define the product

$$z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) \tag{27}$$
$$= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1). \tag{28}$$

Alternatively, if $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

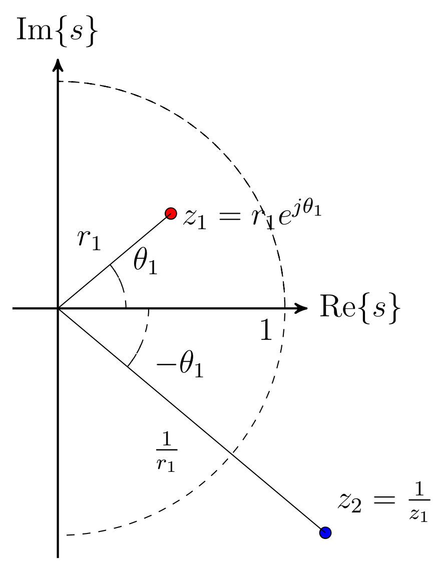


Inversion. If $z = a + jb = re^{j\theta}$, then

$$z^{-1} = \frac{1}{z} = \frac{1}{a+jb} \tag{29}$$
$$= \frac{1}{(a+jb)}\frac{a-jb}{a-jb} \tag{30}$$
$$= \frac{a-jb}{a^2+b^2} \tag{31}$$
$$= \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2}. \tag{32}$$

Alternatively,

$$z^{-1} = \frac{1}{r}e^{-j\theta}.$$



Division. For two numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we define division as multiplication by z_2^{-1}

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(a_1a_2 + b_1b_2) - j(a_1b_2 + a_2b_1)}{a_2^2 + b_2^2}.$$

Alternatively, if $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$$

Definition 2 (Roots Of Complex Polynomials)**.** Let $\alpha(s)$ be a polynomial in the complex variable s, with complex coefficients. If $\alpha(p) = 0$ for $p \in \mathbb{C}$, then p is a *root* of $\alpha(s)$

Definition 3 (Multiplicity)**.** Let p be a root of $\alpha(s)$,

$$\lim_{s\to p} \frac{\alpha(s)}{(s-p)^n} \neq 0, \text{ and}$$

$$\lim_{s\to p} \frac{\alpha(s)}{(s-p)^{n-1}} = 0,$$

then p is a root of $\alpha(s)$ with multiplicity n.

Example 8. Let $\alpha(s) = (s-2)(s-1)^2 s^4$. By our definition above, $p_1 = 2$ is a root of $\alpha(s)$ with multiplicity 1, $p_2 = 1$ is a root with multiplicity 2, and $p_3 = 0$ is a root with multiplicity 4. □

5 Partial Fraction Expansion

The expression $\hat{y}(s)$ for the solution of linear time-invariant (LTI) ODEs, in the s-domain, is the ratio of polynomials in s.

In other words,

$$\hat{y}(s) = \frac{N(s)}{D(s)}.$$

When we want to compute $y(t)$, we need to compute

$$y(t) = \mathcal{L}^{-1}\left\{\hat{y}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{N(s)}{D(s)}\right\}. \tag{33}$$

We use some related ideas to simplify this computation:

- $\mathcal{L}^{-1}\{1/(s-a)\}$ equals e^{at}, when a is real
- any polynomial $a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$ can be rewritten as $a_n(s-p_1)(s-p_2)\cdots(s-p_n)$, where $p_1, \ldots, p_n$ are complex numbers
- For polynomials with real coefficients, if one complex number is a root, its conjugate is always a root.

Loosely speaking, a partial fraction expansion (PFE) of $\hat{y}(s)$ will be of the form

$$\hat{y}(s) = k_0 + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \ldots$$

for complex numbers k_0, k_1, etc. and where $D(s) = (s-p_1)(s-p_2)\cdots(s-p_n)$. Then, $y(t) = \mathcal{L}^{-1}\{\hat{y}(s)\}$ is simply

$$y(t) = k_0\delta(t) + k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \ldots.$$

The expression above doesn't always apply, and we go over the different cases below. In general,

$$\hat{y}(s) = \frac{N_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}, \tag{34}$$

where m is the order of polynomial $N(s)$, n is the order of polynomial $D(s)$, z_i for $i = 1, \ldots, m$ are complex numbers, and p_i for $i = 1, \ldots, n$ are complex numbers.

Some terminology:

1. If $n = m$, then $N(s)/D(s)$ is **exactly proper**.
2. If $n > m$, then $N(s)/D(s)$ is **strictly proper**.
3. The complex numbers z_i are the roots of $N(s)$ and are called **zeros**.
4. The complex numbers p_i are the roots of $D(s)$ and are called **poles**.

The partial fraction expansion of $\hat{y}(s)$ depends on the values of n, m, N_m, z_i, and p_i.

5.1 Case 1: All roots of $D(s)$ are distinct

The PFE of $N(s)/D(s)$ is exactly

$$\hat{y}(s) = k_0 + \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}, \tag{35}$$

where k_0, k_1, $\ldots$, $k_n \in \mathbb{C}$. Furthermore,

$$k_0 = \begin{cases} N_m & \text{if } n = m \\ 0 & \text{if } n > m. \end{cases}$$

Example 9. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{4(s+2)}{(s+1)(s+5)}.$$

Solution:

$$\hat{y}(s) = k_0 + \frac{k_1}{s+1} + \frac{k_2}{s+5} \tag{36}$$

$$= \frac{k_1}{s+1} + \frac{k_2}{s+5} \qquad (n = 2, m = 1, n > m) \tag{37}$$

$$\frac{4(s+2)}{(s+1)(s+5)} = \frac{k_1}{s+1} + \frac{k_2}{s+5} \tag{38}$$

Consider multiplying (38) by $s + 1$:

$$\frac{4(s+2)}{(s+5)} = k_1 + \frac{k_2(s+1)}{s+5}$$

When we set $s = -1$, we get

$$\left.\frac{4(s+2)}{(s+5)}\right|_{s=-1} = k_1 + 0 \implies k_1 = 1$$

Consider multiplying (38) by $s+5$:

$$\frac{4(s+2)}{(s+1)} = k_1 \frac{s+5}{s+1} + k_2$$

When we set $s = -5$, we get

$$\left.\frac{4(s+2)}{(s+1)}\right|_{s=-5} = 0 + k_2. \implies k_2 = 3$$

So,

$$\hat{y}(s) = \frac{1}{s+1} + \frac{3}{s+5} \tag{39}$$

$$\implies y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{3}{s+5}\right) \tag{40}$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) \tag{41}$$

$$= e^{-t} + 3e^{-5t} \tag{42}$$

□

From the previous example, we can identify a **general method for distinct roots**. If

$$\hat{y}(s) = k_0 + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}, \tag{43}$$

then

$$k_i = \left.[\hat{y}(s)(s-p_i)]\right|_{s=p_i}.$$

Example 10. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}.$$

Solution:
We get

$$\hat{y}(s) = k_0 + \frac{k_1}{s+4} + \frac{k_2}{s+2-j} + \frac{k_2}{s+j+2} \tag{44}$$

$$= 3 + \frac{k_1}{s+4} + \frac{k_2}{s+2-j} + \frac{k_2}{s+j+2} \tag{45}$$

To calculate k_1:

$$
\begin{aligned}
k_1 &= \hat{y}(s)(s+4)\big|_{s=-4} && (46)\\
&= \left.\frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}(s+4)\right|_{s=-4} && (47)\\
&= \left.\frac{3(s+1)(s+2)(s+3)}{(s+2-j)(s+j+2)}\right|_{s=-4} && (48)\\
&= \frac{3(-4+1)(-4+2)(-4+3)}{(-4+2-j)(-4+j+2)} && (49)\\
&= \frac{3(-3)(-2)(-1)}{(-2-j)(j-2)} && (50)\\
&= \frac{-18}{5} && (51)
\end{aligned}
$$

To calculate k_2:

$$
\begin{aligned}
k_2 &= \hat{y}(s)(s+2-j)\big|_{s=-2+j} && (52)\\
&= \left.\frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}(s+2-j)\right|_{s=-2+j} && (53)\\
&= \left.\frac{3(s+1)(s+2)(s+3)}{(s+4)(s+j+2)}\right|_{s=-2+j} && (54)\\
&= \frac{3(-2+j+1)(-2+j+2)(-2+j+3)}{(-2+j+4)(-2+j+j+2)} && (55)\\
&= \frac{3(-1+j)(j)(1+j)}{(2+j)(2j)} && (56)\\
&= \frac{-3}{2+j} && (57)\\
&= \frac{-3(2-j)}{5} \qquad \text{(by inversion, Section 4)} && (58)
\end{aligned}
$$

To calculate k_3:

$$\begin{aligned}
k_2 &= \hat{y}(s)(s+2+j)\big|_{s=-2-j} && (59)\\
&= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}(s+2+j)\bigg|_{s=-2-j} && (60)\\
&= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)}\bigg|_{s=-2-j} && (61)\\
&= \frac{3(-2-j+1)(-2-j+2)(-2-j+3)}{(-2-j+4)(-2-j+2-j)} && (62)\\
&= \frac{3(-1-j)(-j)(1-j)}{(2-j)(-2j)} && (63)\\
&= \frac{-3}{2-j} && (64)\\
&= \frac{-3(2+j)}{5} \qquad \text{(by inversion, Section 4)} && (65)
\end{aligned}$$

$$\begin{aligned}
\hat{y}(s) &= k_0 + \frac{k_1}{s+4} + \frac{k_2}{s+2-j} + \frac{k_2}{s+j+2} && (66)\\
&= 3 + \frac{-18}{5(s+4)} + \frac{-3(2-j)}{5(s+2-j)} + \frac{-3(2+j)}{5(s+j+2)} && (67)\\
&= 3 + \frac{-18}{5(s+4)} + \frac{-3}{5}\left(\frac{4s+10}{(s+2)^2+1^2}\right) && (68)
\end{aligned}$$

We combine the last two terms because we will be able to take the inverse Laplace transform of the result. Instead of slogging through the algebra, we can use complex number algebra to handle this step. Notice that if $z = 2-j$, the last two terms are

$$\begin{aligned}
\text{last two terms} &= \frac{-3}{5}\left(\frac{z}{s+z} + \frac{\bar{z}}{s+\bar{z}}\right) && (69)\\
&= \frac{-3}{5}\left(\frac{z(s+\bar{z}) + \bar{z}(s+z)}{(s+z)(s+\bar{z})}\right) && (70)\\
&= \frac{-3}{5}\frac{(z+\bar{z})s + 2z\bar{z}}{(s^2+(\bar{z}+z)s+z\bar{z})} && (71)
\end{aligned}$$

Now, $z+\bar{z} = 2\text{Re}\{z\} = 2\cdot 2$, and $z\bar{z} = |z|^2 = 2^2+1^2 = 5$. Therefore, we get

$$\text{last two terms} = \frac{-3}{5}\left(\frac{4s+10}{s^2+4s+5}\right).$$

This looks a little nicer, in part because

$$\mathcal{L}^{-1}\left\{\frac{(s+a)}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt, \text{ and}$$

$$\mathcal{L}^{-1}\left\{\frac{c}{(s+a)^2+b^2}\right\} = \frac{c}{b}e^{-at}\sin bt.$$

and we will be able to apply this rule. The first step is to simplify the denominator, by completing squares:

$$s^2+4s+5 \to s^2+4s+4+1 \to (s+2)^2+1^2.$$

This step also tells us how to modify the numerator:

$$4s+10 \to 4(s+2-2)+10 \to 4(s+2)+10-8 \to 4(s+2)+2.$$

We now get the last two terms into the form

$$\text{last two terms} = \frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right).$$

We are now ready to take the inverse of

$$\hat{y}(s) = 3 + \frac{-18}{5(s+4)} + \frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right). \tag{72}$$

$$\mathcal{L}^{-1}\{\hat{y}(s)\} = \mathcal{L}^{-1}\left\{3 + \frac{-18}{5(s+4)} + \frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right)\right\} \tag{73}$$

$$= \mathcal{L}^{-1}\{3\} + \mathcal{L}^{-1}\left\{\frac{-3.6}{(s+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right)\right\} \tag{74}$$

$$= 3\delta(t) - 3.6e^{-4t} + \mathcal{L}^{-1}\left\{\frac{-3}{5}\left(\frac{4(s+2)}{(s+2)^2+1^2}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{5}\left(\frac{2}{(s+2)^2+1^2}\right)\right\} \tag{75}$$

$$= 3\delta(t) - 3.6e^{-4t} + \mathcal{L}^{-1}\left\{\left(\frac{-2.4(s+2)}{(s+2)^2+1^2}\right)\right\} + \mathcal{L}^{-1}\left\{-1.2\left(\frac{1}{(s+2)^2+1^2}\right)\right\} \tag{76}$$

$$= 3\delta(t) - 3.6e^{-4t} - 2.4e^{-2t}\cos t - 1.2e^{-2t}\sin t, \tag{77}$$

Which is the solution to Example 10. □

5.2 Case 2: Roots of $D(s)$ are repeated

$$\hat{y}(s) = \frac{N_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)^{l_1}(s-p_2)^{l_2}\cdots(s-p_q)^{l_q}}, \tag{78}$$

where $n = l_1 + l_2 + \cdots l_q$. The PFE in this case is

$$\begin{aligned}\hat{y}(s) = k_0 &+ \frac{k_1}{(s-p_1)^{l_1}} + \frac{k_2}{(s-p_1)^{l_1-1}} + \cdots + \frac{k_{l_1}}{(s-p_1)} + \frac{k_{l_1+1}}{(s-p_2)^{l_2}} \\ &+ \cdots + \frac{k_{n-l_q}}{(s-p_q)^{l_q}} + \frac{k_{n-l_q+1}}{(s-p_q)^{l_q-1}} + \cdots + \frac{k_n}{s-p_q}.\end{aligned} \tag{79}$$

where k_0, k_1, …, $k_n \in \mathbb{C}$. Again,

$$k_0 = \begin{cases} N_m & \text{if } n = m \\ 0 & \text{if } n > m. \end{cases}$$

Example 11. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{1}{(s+2)(s+1)^2}.$$

Solution: The roots are: $p_1 = -2$ with multiplicity 1, and $p_2 = -1$ with multiplicity 2. Therefore.

$$\hat{y}(s) = k_0 + \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{80}$$

$$= \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_2}{(s+1)^2} \qquad (n = 3, m = 0, n > m) \tag{81}$$

Since p_1 has multiplicity 1, we can obtain k_1 using the same rule as for distinct roots:

$$k_1 = \hat{y}(s)(s+2)\big|_{s=-2} \tag{82}$$

$$= \frac{1}{(s+2)(s+1)^2}(s+2)\Big|_{s=-2} \tag{83}$$

$$= \frac{1}{(s+1)^2}\Big|_{s=-2} \tag{84}$$

$$= \frac{1}{(-2+1)^2} \tag{85}$$

$$= 1 \tag{86}$$

(87)

This rule works for distinct roots p_i because we know all other terms have to go to zero when evaluating at $s = p_i$. When we have a root p_j with multiplicity greater than 1, multiplying by $(s - p_j)$ won't work. Let's see why:

$$\frac{1}{(s+2)(s+1)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{88}$$

$$\implies \frac{1}{(s+2)(s+1)^2}(s+1) = \frac{k_1}{s+2}(s+1) + \frac{k_2}{s+1}(s+1) + \frac{k_3}{(s+1)^2}(s+1) \tag{89}$$

$$\implies \frac{1}{(s+2)(s+1)} = \frac{k_1(s+1)}{s+2} + k_2 + \frac{k_3}{(s+1)} \tag{90}$$

We can't plug in $s = -1$, so that the following equation suggested by Equation (81) is incorrect:

$$\text{Incorrect:} \quad k_2 = \hat{y}(s)(s+1)\big|_{s=-1} .$$

As you might guess, the only thing that makes sense is multiplying by $(s - p_j)^l$, where l is the multiplicity of root p_j. In our example:

$$\frac{1}{(s+2)(s+1)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{91}$$

$$\implies \frac{1}{(s+2)(s+1)^2}(s+1)^2 = \frac{k_1}{s+2}(s+1)^2 + \frac{k_2}{s+1}(s+1)^2 + \frac{k_3}{(s+1)^2}(s+1)^2 \tag{92}$$

$$\implies \frac{1}{(s+2)} = \frac{(s+1)^2}{s+2} + k_2(s+1) + k_3 \tag{93}$$

If $s = -1$, the only terms remaining are k_3 and the left hand side which is $\hat{y}(s)(s+1)^2$. It turns out that we could have used the same pattern as in the case of distinct roots only for the term containing the $(s - p_j)^l$, which here is k_3:

$$\text{Correct:} \quad k_3 = \hat{y}(s)(s+1)^2\big|_{s=-1} . = \left.\frac{1}{s+2}\right|_{s=-1} = \frac{1}{-1+2} = 1$$

In other words, we can use the following more general rule: If the PFE of $\hat{y}(s)$ contains the term $k_i/(s - p_j)^l$, then

$$k_i = \hat{y}(s)(s-p_j)^l\big|_{s=p_j} , \text{ only when } l \text{ is the multiplicity of pole } p_j.$$

This rule includes the case of poles with multiplicity 1.

What about terms of the form $k_i/(s - p_j)^{l'}$, where l' is **less** than the multiplicity l of p_j? First, note that we would expect $l-1$ such terms, as defined in the PFE (79) for the repeated root case. We use the following approach:

1. Multiply the expression involving the PFE by $(s - p_j)^l$, where l is the multiplicity of pole p_j.

2. Differentiate the expression with respect to s, a total of $l-1$ times, using the expression after each time you differentiate to calculate one of the $l-1$ coefficients by plugging in $s = p_j$.

So, in our still running example:

$$\frac{1}{(s+2)(s+1)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{94}$$

$$\implies \frac{1}{(s+2)} = \frac{(s+1)^2}{s+2} + k_2(s+1) + k_3 \qquad \text{(Multiplying by } (s+1)^2 \text{)} \tag{95}$$

Notice that when we substitute in $s = -1$, on the right hand side only the coefficient in front of the term without $(s+1)$ remains. How do we make that coefficient be k_2? The easy way is to differentiate. This does two things: 1) k_3 disappears 2) the terms with higher powers of $(s+1)$ will still contain $(s+1)$, and so we don't have to explicitly evaluate the derivative:

$$\frac{d}{ds}\left(\frac{1}{(s+2)} = \frac{(s+1)^2}{s+2} + k_2(s+1) + k_3\right) \tag{96}$$

$$\implies \frac{-1}{(s+2)^2} = \frac{d}{ds}\left(\frac{(s+1)^2}{s+2}\right) + k_2 + 0 \tag{97}$$

Again, we don't worry about the first term on the RHS for now because it evaluates to 0 when we plug in $s = -1$. So, let's plug in $s = -1$

$$\frac{-1}{(-1+2)^2} = 0 + k_2 \implies k_2 = -1.$$

So, we have now completed the PFE.

$$\hat{y}(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2} \tag{98}$$

Let's take the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\{\hat{y}(s)\} \tag{99}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}\right\} \tag{100}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \tag{101}$$

$$= e^{-2t} - e^{-t} + te^{-t} \qquad \text{(by the } s\text{-shift and multiplication-by-time rules),} \tag{102}$$

which is the solution to Example 11 □

5.3 Summary of Partial Fraction Expansions

If there are m roots, no matter what their multiplicities are, we will be able to obtain m coefficients in the PFE by direct calculation. The PFE contains the term $k_i/(s-p_j)^l$ where l is the multiplicity of p_j, and

$$k_i = \hat{y}(s)(s-p_j)^l\big|_{s=p_j}.$$

For the terms of the form $k_i/(s-p_j)^{l'}$, where l' is less than the multiplicity of p_j, we do the following:

1. Multiply the expression involving the PFE of $\hat{y}(s)$ by $(s-p_j)^l$, where l is the multiplicity of pole p_j.
2. Differentiate the expression with respect to s, a total of $l-1$ times, using the expression after each time you differentiate to calculate one of the $l-1$ coefficients by plugging in $s = p_j$.

Example 12. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{3(s+2)(s+1)}{(s+5)^3}.$$

Solution steps:

1. Calculate the poles
2. Write down the form of the PFE, containing unknown coefficients
3. Use n and m to calculate k_0
4. Calculate coefficients for term corresponding to highest multiplicity of pole directly
5. Calculate the remaining coefficient corresponding to repeated roots using differentiation
6. Express $\hat{y}(s)$ using the calculated coefficients
7. Calculate $y(t)$ using the inverse Laplace transform

The roots are: $p_1 = -5$ with multiplicity 3. Therefore

$$\hat{y}(s) = k_0 + \frac{k_1}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_3}{(s+5)}. \tag{103}$$

Since $n = 3$ and $n = 2$, $k_0 = 0$. Therefore,

$$\hat{y}(s) = \frac{k_1}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_3}{(s+5)}. \tag{104}$$

We can calculate k_1 directly:

$$k_1 = \hat{y}(s)(s+5)^3\big|_{s=-5} \tag{105}$$
$$= \frac{3(s+2)(s+1)}{(s+5)^3}(s+5)^3\bigg|_{s=-5} \tag{106}$$
$$= 3(s+2)(s+1)|_{s=-5} \tag{107}$$
$$= 3(s+2)(s+1)|_{s=-5} \tag{108}$$
$$= 3(-5+2)(-5+1) \tag{109}$$
$$= 36 \tag{110}$$

To get k_2 and k_3, first multiply the PFE by $(s+5)^3$

$$3(s+2)(s+1) = k_1 + k_2(s+5) + k_3(s+5)^2.$$

Differentiate with respect to s

$$3(s+2) + 3(s+1) = 0 + k_2 + k_3 2(s+5) \tag{111}$$

Set $s = -5$, to get

$$3(-5+2) + 3(-5+1) = k_2 + 0k_3 \implies k_2 = -21.$$

Differentiate (122) with respect to s

$$3 + 3 = 0 + 0 + 2k_3 \tag{112}$$

We 'plug in' $s = -5$ into (123) gives $k_3 = 3$.

So,

$$\hat{y}(s) = \frac{36}{(s+5)^3} - \frac{21}{(s+5)^2} + \frac{3}{(s+5)}, \text{ and} \tag{113}$$

$$y(t) = \mathcal{L}^{-1}\{\hat{y}(s)\} \tag{114}$$
$$= \mathcal{L}^{-1}\left\{\frac{36}{(s+5)^3} - \frac{21}{(s+5)^2} + \frac{3}{(s+5)}\right\} \tag{115}$$
$$= \mathcal{L}^{-1}\left\{\frac{36}{(s+5)^3}\right\} - \mathcal{L}^{-1}\left\{\frac{21}{(s+5)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+5)}\right\} \tag{116}$$
$$= 18t^2e^{-5t} - 21te^{-5t} + 3e^{-5t}, \quad \text{(using the multiplication-by-}t^n\text{ rule)}, \tag{117}$$

□

Example 13. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{3(s+2)(s+1)}{(s+5)^3}.$$

Solution:

$$\hat{y}(s) = k_0 + \frac{k_1}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_3}{(s+5)}. \tag{118}$$

$n = 3$, $n = 2$. $k_0 = 0$.

$$\hat{y}(s) = \frac{k_1}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_3}{(s+5)}. \tag{119}$$

$$k_1 = \hat{y}(s)(s+5)^3\big|_{s=-5} \tag{120}$$
$$= 36 \tag{121}$$

k_2 :

$$3(s+2)(s+1) = k_1 + k_2(s+5) + k_3(s+5)^2.$$

$$3(s+2) + 3(s+1) = 0 + k_2 + k_3 2(s+5) \tag{122}$$

$$3(-5+2) + 3(-5+1) = k_2 + 0k_3 \implies k_2 = -21.$$

k_3:

$$3 + 3 = 0 + 0 + 2k_3 \tag{123}$$

$\implies k_3 = 3.$

$$\hat{y}(s) = \frac{36}{(s+5)^3} - \frac{21}{(s+5)^2} + \frac{3}{(s+5)}, \text{ and} \tag{124}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{36}{(s+5)^3} - \frac{21}{(s+5)^2} + \frac{3}{(s+5)}\right\} \tag{125}$$
$$= 18t^2e^{-5t} - 21te^{-5t} + 3e^{-5t}, \tag{126}$$

□