Transfer Functions

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Contents

1 Introduction

Consider a dynamical system with input-output ODE

$$\begin{aligned} &a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) \\ &\quad = b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t). \end{aligned} \tag{1}$$

We can view this dynamical system as an object that converts one function of time (the input) into another function of time (the output):

Figure 1: A dynamical system converts a function of time (input $u(t)$) into another function of time (output $y(t)$).

Calculating the output function due to a given input function is the same as solving the ODE. We have seen how to use Laplace transforms, and their inverse, to calculate the solution to the differential equation given initial conditions $y^{(n-1)}(t_0)$, $y^{(n-2)}(t_0)$, $\ldots$, $y(t_0)$, and some input $u(t)$.

Given a dynamical system, we would like to not have to compute the solution of (1) for all possible initial conditions and inputs. Is there a way to predict how the solutions will behave, without exhaustively simulating every possible situation? Transfer functions allow us to perform such a prediction.

2 Deriving Transfer Functions

Definition 1. The transfer function from input to output is obtained from the Laplace transform of the input-output differential equation when all initial conditions are zero.

Let's apply this definition. Due to the linearity of the Laplace transform, we see that we will have to calculate terms of the form $a_n L\{y^{(n)}(t)\}$, which we know equals

$$\begin{aligned} L\{y^{(n)}(t)\} &= s^n \hat{y}(s) - s^{n-1} y(0) - s^{n-2} \dot{y}(0) - s^{n-3} \ddot{y}(0) - \cdots - s y^{(n-2)}(0) - y^{(n-1)}(0) \\ &= s^n \hat{y}(s) - \mathrm{ICT_y}, \end{aligned}$$

where $\mathrm{ICT_y}$ refers to terms depending on the initial conditions of y.

Therefore, the Laplace transform applied to (1) gets us

$$
\begin{aligned}
&a_n s^n \hat{y}(s) + a_{n-1} s^{n-1} \hat{y}(s) + \cdots + a_2 s^2 \hat{y}(s) + a_1 \hat{y}(s) + a_0 \hat{y}(s) - \mathrm{ICT_y} \\
&\quad = b_m s^m \hat{u}(s) + b_{m-1} s^{m-1} \hat{u}(s) + \cdots + b_2 s^2 \hat{u}(s) + b_1 s \hat{u}(s) + b_0 \hat{u}(s) - \mathrm{ICT_u}.
\end{aligned} \tag{2}
$$

Define

$$
\begin{aligned}
\alpha(s) &= a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0 \\
\beta(s) &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_2 s^2 + b_1 s + b_0.
\end{aligned}
$$

We then rewrite (2) as

$$
\hat{y}(s) = \frac{\beta(s)}{\alpha(s)} \hat{u}(s) - \frac{1}{\alpha(s)} \mathrm{ICT_u} + \frac{1}{\alpha(s)} \mathrm{ICT_y}. \tag{3}
$$

$$
= \underbrace{G(s)\hat{u}(s) - \frac{1}{\alpha(s)} \mathrm{ICT_u}}_{\text{Forced response}} + \underbrace{\frac{1}{\alpha(s)} \mathrm{ICT_y}}_{\text{Free response}}. \tag{4}
$$

The quantity $G(s) = \beta(s)/\alpha(s)$ is known as the **transfer function** of the dynamical system. The first two terms correspond to the **forced response**, and the last term to the **free response**.

We can redraw Figure 1 as

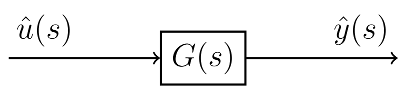


Figure 2: A dynamical system converts input $\hat{u}(s)$ into output $\hat{y}(s)$.

If $\mathrm{ICT_u} = 0$ and $\mathrm{ICT_y} = 0$, then $\hat{y}(s) = G(s)\hat{u}(s)$. The main idea is that studying $G(s)$ by itself may tell us what we need to know about response $y(t) = L^{-1}\{G(s)\hat{u}(s)\}$ for some types of inputs $\hat{u}(s)$. **In other words, $G(s)$ can tell us useful things about all possible forced responses that a set of inputs may produce, without having to explicitly solve the ODE for each one.**

Note that if $u(t) = \delta(t)$, then $\hat{u}(s) = 1$, so that $\hat{y}(s) = G(s)$. Therefore, $G(s)$ is often also called the impulse response function of the dynamical system.

To summarize, given a linear time-invariant ODE

$$
\begin{aligned}
&a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) \\
&\quad = b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t).,
\end{aligned}
$$

its transfer function $G(s)$ is

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_2 s^2 + b_1 + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 + a_0} = \frac{\beta(s)}{\alpha(s)}. \tag{5}$$

This transfer function, for linear time invariant (LTI) systems, is a rational function of the complex variable s. In other words, it is the ratio of two polynomials in s with real coefficients.

3 Classification of Transfer Functions

Definition 2 (Order Of A Transfer Function)**.** The order of a rational transfer function $G(s)$ is the degree of the polynomial in the denominator of $G(s)$. For example, the order of $G(s)$ in (5) is n.

Definition 3. If $n = m$, then $G(s)$ is exactly proper.

Definition 4. If $n > m$, then $G(s)$ is strictly proper.

Definition 5. The relative degree of $G(s)$ is $n - m$.

Definition 6. The poles of $G(s)$ are the roots of $\alpha(s)$.

Definition 7. The zeros of $G(s)$ are the roots of $\beta(s)$.

4 Poles and Zeros

Consider the transfer function $G(s)$ in (5). The polynomial $\beta(s)$ has degree m, and the polynomial $\alpha(s)$ has degree n, which is also the degree of $G(s)$. We define the poles and zeros of $G(s)$ as follows

Definition 8 (Pole of a Transfer Function)**.** A complex number $p \in \mathbb{C}$ is a pole of $G(s) = \beta(s)/\alpha(s)$ if $\alpha(p) = 0$.

Definition 9 (Zero of a Transfer Function)**.** A complex number $z \in \mathbb{C}$ is a zero of $G(s) = \beta(s)/\alpha(s)$ if $\beta(z) = 0$.

We can express a transfer function as the ratio of two polynomials:

$$G(s) = \frac{\beta(s)}{\alpha(s)} = \frac{N_m(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}, \tag{6}$$

where p_i and z_i are respectively the poles and zeros of $G(s)$.

4.1 Pole-Zero Maps

The poles and zeros are complex numbers, and we can plot their location in the complex plane. Such a plot is known as a **pole-zero map**.

There are five zones or regions of the complex plane that have special significance. These are the Imaginary Axis (IA), the Open Left Half Plane (OLHP), the Open Right Half Plane (ORHP), the Closed Left Half Plane (CLHP), and the Closed Right Half Plane (CRHP). See Table 1 for definitions, and Figure 3 for a diagram.

Region	Abbr.	Definition	Notes
Imaginary Axis	IA	$\text{Re}(s) = 0$	
Open Left Half Plane	OLHP	$\text{Re}(s) < 0$	
Open Right Half Plane	ORHP	$\text{Re}(s) > 0$	
Closed Left Half Plane	CLHP	$\text{Re}(s) \leq 0$	OLHP $\cup$ IA
Closed Right Half Plane	CRHP	$\text{Re}(s) \geq 0$	ORHP $\cup$ IA

Table 1: Regions of the complex plane.

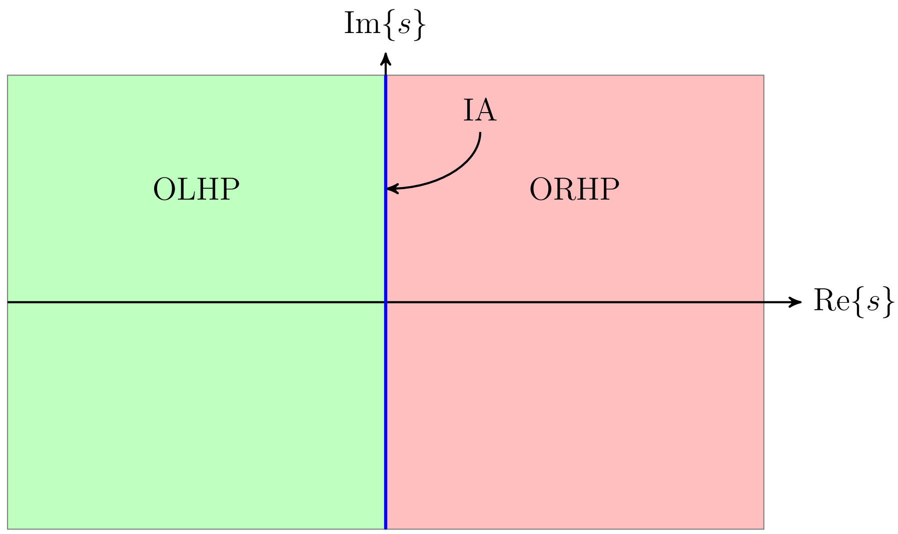


Figure 3: The OLHP (light green region), ORHP (light red region), and Imaginary Axis (blue line). These three sets have NO common points. The IA separates the OLHP and ORHP, acting as a boundary to both sets. The CLOSED left half plane is the OLHP together with its boundary which is the IA. Similarly, the closed RIGHT half plane is the ORHP together with the IA.

The words **open** and **closed** have precise mathematical meanings from set topology. The imaginary axis is a (vertical) line that forms the boundary of both right and left halves of the complex plane. A closed set contains its boundary. Therefore, the **closed right** half plane contains all points to the **right** of the imaginary axis **and also** the points **on** the imaginary axis. Similarly, the **closed left** half plane contains all points to the **left** of the imaginary axis **and also** the points **on** the imaginary axis.

An **open** set is one that does **not** contain its boundary. The **open left** half plane contains all points to the **left** of the imaginary axis, and does **not** contain any points on the imaginary axis. A corresponding description holds for the ORHP.

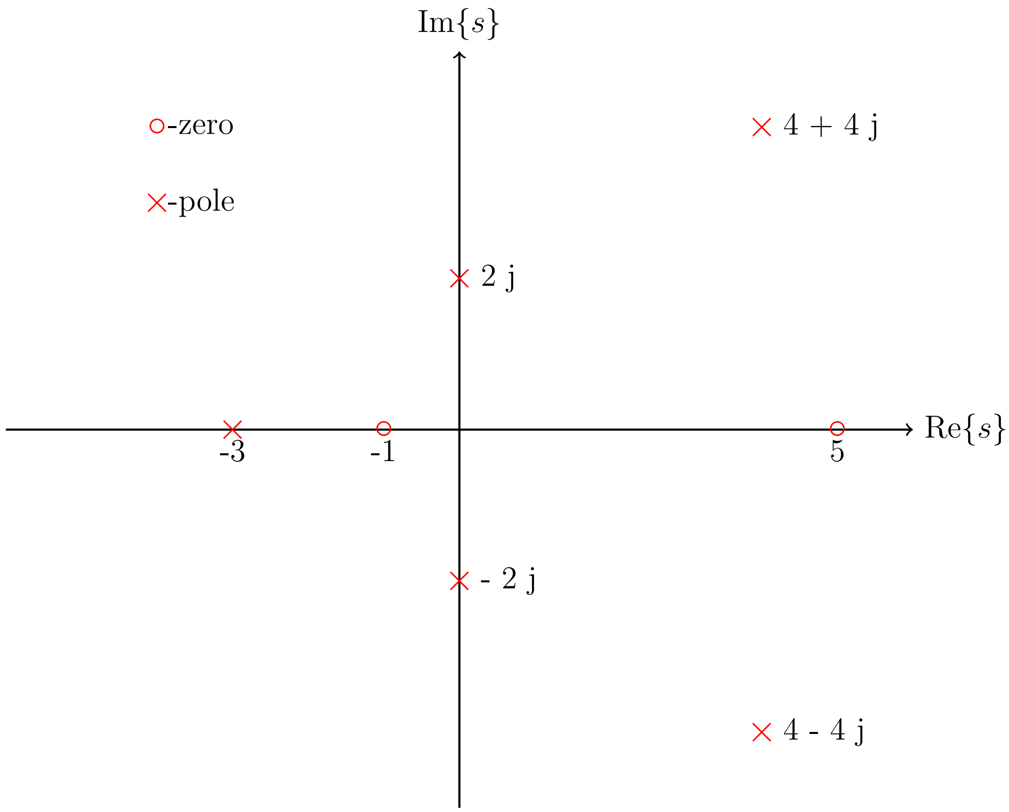
All these statements lead to some equations involving sets as variables:

1. ORHP, OLHP, and IA are distinct sets:
   (a) $\text{ORHP} \cap \text{OLHP} = \emptyset$
   (b) $\text{ORHP} \cap \text{IA} = \emptyset$
   (c) $\text{OLHP} \cap \text{IA} = \emptyset$
2. Together, they form $\mathbb{C}$:
   $\mathbb{C} = \text{ORHP} \cup \text{OLHP} \cup \text{IA}$
3. The imaginary axis is part of both closed regions:
   (a) $\text{CLRP} \cap \text{CHRP} = \text{IA}$
   (b) $\text{IA} \subset \text{CRHP}$
   (c) $\text{IA} \subset \text{CLHP}$
4. An open region combined with the IA forms a closed region:
   (a) $\text{CHRP} = \text{IA} \cup \text{ORHP}$, also $\text{ORHP} = \text{CRHP} - \text{IA}$
   (b) $\text{CLRP} = \text{IA} \cup \text{OLHP}$, also $\text{OLHP} = \text{CLHP} - \text{IA}$

Example 1. Consider the transfer function

$$G(s) = \frac{5(s+1)(s-5)}{(s+3)(s-4-4j)(s-4+4j)(s^2+4)}. \tag{7}$$

Its pole-zero map is:



Let's describe the regions these poles and zeros belong to:

1. The poles $\pm 2j$ are on the imaginary axis. Therefore, they are also belong to the CLHP and CRHP. However, they are NOT in either the ORHP or the OLHP.
2. The poles $4 \pm 4j$ are in the ORHP, which automatically make them part of the CRHP (but not the IA).
3. Likewise, the zero 5 is in the ORHP and CRHP (but not IA).
4. The zero -1 and pole -3 are in the OLHP, and therefore also in the CLHP, but they are not on the IA.

5 System Responses

Recall that the response $\hat{y}(s)$ of a system $G(s)$ to an input $\hat{u}(s)$ is $G(s)\hat{u}(s)$.

As mentioned in Section 2, we may be able to characterize the response $y(t)$ generated by a type of input $\hat{u}(s)$ by looking at $G(s)$, instead of solving for $y(t) = \mathcal{L}^{-1}\left\{G(s)\hat{u}(s)\right\}$.

This section describes some of these characteristics.

6 Stability Of A System

Let $G(s)$ be the transfer function of a linear time invariant (LTI) system (1).

Typically, the case where $y(t) = 0$ is an equilibrium. For example, consider the simple pendulum at its downward position, with zero velocity. A quick sidewards tap on the mass is known as providing an *impulse*. We're sure that the pendulum moves away from the downward position in response to the tap. But what will happen in the long run? Here, we're asking for the **impulse response** of the simple pendulum.

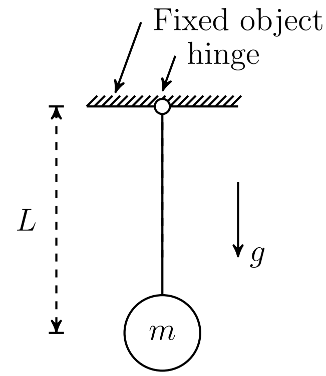


Figure 4: Pendulum in downward equilibrium position.

For any input or initial conditions, there are three possible behaviors of the impulse response $y_i(t) = \mathcal{L}^{-1}\{G(s)\}$:

1. $y_i(t)$ is unbounded ($|y_i(t)| \to \infty$)
2. $y_i(t)$ is bounded (We can find $0 < M < \infty$ such that $|y_i(t)| \leq M$ for all t)
3. $\lim_{t\to\infty} y(t) = 0$

We can use these three behaviors to define three notions of stability:

Definition 10 (Unstable)**.** $G(s)$ is unstable (US) if its impulse response is unbounded.

Definition 11 (Lyapunov Stable)**.** $G(s)$ is Lyapunov stable (LS) if its impulse response is bounded.

Definition 12 (Asymptotically Stable)**.** $G(s)$ is asymptotically stable (AS) if its impulse response satisfies $\lim_{t\to\infty} y_i(t) = 0$.

Note: An asymptotically stable TF is Lyapunov stable. An unstable system is not LS, and therefore not AS either.

Remark: Why are we interested in $y_i(t) \to 0$ instead of $y_i(t) \to a$, where $a \neq 0$? The answer is that we assume we are interested in equilibria, and for a linear system, 0 is its equilibrium.

6.1 Stability and Poles of the Transfer Function

Let's apply the notion of multiplicity of roots, first mentioned in Laplace transforms, to the multiplicity of poles.

Definition 13 (Multiplicity)**.** Let $G(s) = \beta(s)/\alpha(s)$. If $\alpha(p) = 0$,

$$\lim_{s \to p} \frac{\alpha(s)}{(s-p)^n} \neq 0, \text{ and}$$

$$\lim_{s \to p} \frac{\alpha(s)}{(s-p)^{n-1}} = 0,$$

then p is a pole of $G(s)$ with multiplicity n.

Example 2. Let

$$G(s) = \frac{1}{(s-2)(s-1)^2 s^3}.$$

Now, $\alpha(s) = (s-2)(s-1)^2 s^3$. By our definition above, $p_1 = 2$ is a pole of $G(s)$ with multiplicity 1, $p_2 = 1$ is a pole with multiplicity 2, and $p_3 = 0$ is a pole with multiplicity 3.

Fact: $G(s)$ is AS if and only if all its poles are in the open left half plane (OLHP).

Fact: $G(s)$ is LS if all its poles either

- are in the OLHP, or
- are on the imaginary axis (IA) with multiplicity one.

Fact: G(s) is US if has a pole

- in the open right half plane (OHRP), or
- on the imaginary axis (IA) with multiplicity greater than 1.

These facts above are one example of the statement made in Section 2 where studying $G(s)$ tells us something about the responses of a system to a given set of inputs.

Example 3. Consider a system with transfer function $G(s)$ given by

$$G(s) = \frac{s}{s^2 + 5s + 6}. \tag{8}$$

Classify the stability properties of this sytem.

Solution: The denominator polynomial is $s^2 + 5s + 6$. The poles of $G(s)$ are therefore the solutions (or, roots) of the equation $s^2 + 5s + 6 = 0$, which turn out to be $p_1 = -2, p_2 = -3$ (we could also have said $p_1 = -3, p_2 = -2$). The real part of both these roots are in the OLHP, therefore $G(s)$ is asymptotically stable. Since $G(s)$ is asymptotically stable (AS), it is also Lyapunov stable (LS).

Example 4. Consider a system with transfer function $G(s)$ given by

$$G(s) = \frac{s}{s^2 - 6s + 5}. \tag{9}$$

Classify the stability properties of this sytem.

Solution: The denominator polynomial is $s^2 - 6s + 5$. The poles of $G(s)$ are therefore the solutions (or, roots) of the equation $s^2 - 6s + 5 = 0$, which turn out to be the complex conjugate pair $p_{1,2} = 3 \pm j2$. The real part of both these roots are strictly positive, the poles are in the ORHP, therefore $G(s)$ is unstable (US).

6.2 Initial and Final Value Theorems

In some case, we may only want to know the value of $y(t)$ at specific times of interest, and solving for $y(t)$ using the inverse Laplace transform is involved. For example, consider a system that is stable, but not asymptotically stable. Then, $y(t)$ remains bounded, and if it approaches a constant value, we'd like to know what that value is. We may be able to calculate this value without ever solving for $y(t)$. The Final Value Theorem helps us do this.

6.2.1 Final Value Theorem

Let $y(t)$ have the Laplace transform $\hat{y}(s)$ (which could be a response of the form $G(s)\hat{u}(s)$). If the poles of $\hat{y}(s)$ are in the OLHP with the possible exception of a single pole at zero. Then,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s). \tag{10}$$

Proof: Use the Laplace transform of $\dot{y}$ along with the Laplace transform of the derivative.

6.2.2 Initial Value Theorem

A similar result let's us know what the initial value is, but there are no conditions on the poles of $G(s)$

$$y(0) = \lim_{t\to 0} y(t) = \lim_{s\to\infty} s\hat{y}(s). \tag{11}$$

Example 5. Consider the response $y_{step}(t)$ of a second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

to a step input $\hat{u}_{step}(s) = 1/s$. to a step input:

$$\hat{y}_{step} = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)},$$

where $\xi > 0$ Find the initial and final value of the response.

Solution:
Initial Value:

$$\begin{aligned} y_{step}(0) &= \lim_{s\to\infty} s\hat{y}(s) && (12) \\ &= \lim_{s\to\infty} s\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} && (13) \\ &= \lim_{s\to\infty} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} && (14) \\ &= 0 && (15) \end{aligned}$$

Final Value: Since $\xi > 0$, two poles of

$$\hat{y}_{step} = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)},$$

are in the open left half plane, and one on the imaginary axis. Therefore, we may use the FVT to calcluate $y_{step}(\infty)$.

$$\begin{aligned} y_{step}(\infty) &= \lim_{s\to 0} s\hat{y}(s) && (16) \\ &= \lim_{s\to 0} s\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} && (17) \\ &= \lim_{s\to 0} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} && (18) \\ &= 1 && (19) \end{aligned}$$

Together, we learn that the system $G(s)$ will have an output in response to a step input that starts from zero, and reaches 1, meaning that the ouput eventually matches the input. This property is very useful for reference-tracking systems, like in servo motors.

Example 6. If we have a mass-spring-damper at equilibrium, where

$$G(s) = \frac{1}{ms^2 + cs + k},$$

and apply a step input force $f(t)$ on the mass, the response of the position of the mass $y = q$ is

$$\hat{y}(s) = \frac{1}{(ms^2 + cs + k)} \frac{1}{s} \implies y_{step}(\infty) = \frac{1}{k} \tag{20}$$

We've just shown that a stiffer spring (higher k) reduces the distance (smaller $1/k$) by which a constant force (step input $f(t)$) moves the resting position $(0 \to \frac{1}{k})$.