Linearization

Hasan A. Poonawala
Fall 2020

Contents

1 Introduction

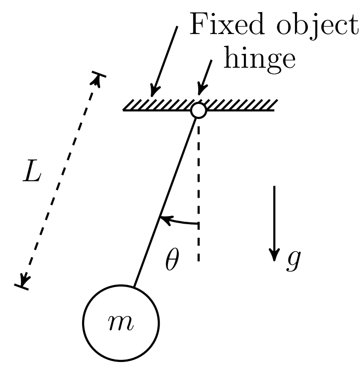
Most systems in the world are nonlinear. Many nonlinear systems, however, can be approximated by linear systems.

We refer to this process of approximating a nonlinear system by a linear system as **linearization**.

Example 1 (Simple Pendulum)**.** The simple pendulum is a mass m suspended by a rigid massless string of length L from a point, moving under the effect of gravity g. This system can be described by the angle θ, with time t as the independent variable. The rotational version of Newton's laws provide the EoM:

$$mL^2\ddot{\theta} + mgL\sin\theta = \tau(t), \qquad (1)$$

where $\tau(t)$ is an input torque applied at the hinge. □



It may be hard to solve (1) given an initial conditon $\theta(0)$ and $\dot{\theta}(0)$, and an input $\tau(t)$. However, a careful combination of engineering judgement and mathematical tools will allow us to understand the behavior of the simple pendulum in some cases.

In previous material, we saw that it is possible to analyze linear systems. Through transfer functions, we can say a great deal about how linear systems respond to inputs. Therefore, we will analyze a nonlinear system like the one in (1) by approximating it using a linear system. The main idea is to **linearize the system around an equilibrium**. This concept is related to relative displacements, and uses the Taylor Expansion of an analytic function.

2 Taylor Series Expansion

The linearization process uses the Taylor expansion of a function $f(x)$ about a point x_e. For this process to work well, the function $f(x)$ must be differentiable arbitrarily many times. We distinguish between two cases:

1. Single variable $f\colon \mathbb{R} \to \mathbb{R}$
2. Multi-variable $f\colon \mathbb{R}^n \to \mathbb{R}$

2.1 Single Variable Function

The Taylor expansion of f about a point x_e is

$$f(x) = f(x_e) + \frac{\mathrm{d}}{\mathrm{d}x} f(x)\bigg|_{x=x_e} (x - x_e) + \frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}^2 x} f(x)\bigg|_{x=x_e} (x - x_e)^2 + \frac{1}{2\cdot 3}\,\frac{\mathrm{d}^3}{\mathrm{d}^3 x} f(x)\bigg|_{x=x_e} (x - x_e)^3 + \cdots \tag{2}$$

$$= f(x_e) + \sum_{k=1}^{\infty} \frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}^k x} f(x)\bigg|_{x=x_e} (x - x_e)^k \tag{3}$$

If $x - x_e$ is small, then $(x - x_e)^k$ for $k \geq 2$ will be smaller, and so the approximation $\hat{f}(x)$ of $f(x)$ is

$$\hat{f}(x) \approx f(x_e) + \frac{\mathrm{d}}{\mathrm{d}x} f(x)\bigg|_{x=x_e} (x - x_e),$$

which is exactly the linear (aka first-order) approximation of f about x_e.

To see why this is a linear approximation, let's think in terms of deviations of x from x_e and corresponding deviations of $f(x)$ from $f(x_e)$. Let $\delta x = x - x_e$ be the local deviation (perturbation) from x_e. Further, if we define $y = f(x)$, $y_e = f(x_e)$, and $\delta y = y - y_e$, then,

$$f(x) - f(x_e) \approx \frac{\mathrm{d}}{\mathrm{d}x} f(x)\bigg|_{x=x_e} (x - x_e), \quad \text{or}$$

$$\delta y \approx f'(x_e)\delta x,$$

which is a linear relationship between δy and δx.

What would this linearization look like?

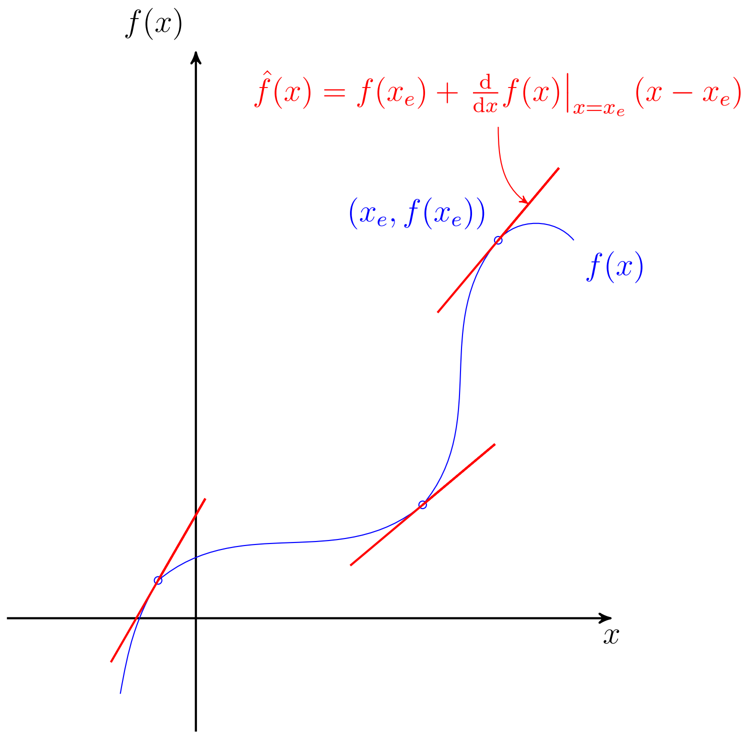


Figure 1: Approximation of a function $f(x)$ at x_e through linearization.

Example 2 (Small Angle Approximation)**.** Find the Taylor-series exapansion and first-order approximation of $\sin x$ about $x_e = 0$.

Solution:

$$\begin{aligned}
\sin x &= \sin x_e + \sum_{k=1}^{\infty} \frac{\mathrm{d^k}}{\mathrm{d^k}x}(\sin x)\bigg|_{x=x_e} (x - x_e)^k \\
&= \sin(0) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\bigg|_{x=0} (x-0) + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}^2x}(\sin x)\bigg|_{x=0} (x-0)^2 + \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}^3x}(\sin x)\bigg|_{x=0} (x-0)^3 \\
&\quad + \frac{1}{4!}\frac{\mathrm{d}^4}{\mathrm{d}^4x}(\sin x)\bigg|_{x=0} (x-0)^4 + \frac{1}{5!}\frac{\mathrm{d}^5}{\mathrm{d}^5x}(\sin x)\bigg|_{x=0} (x-0)^5 + \cdots \\
&= 0 + (\cos 0)x + \frac{(-\sin 0)}{2}x^2 + \frac{(-\cos 0)}{3!}x^3 + \frac{(\sin 0)}{4!}x^4 + \frac{(\cos 0)}{5!}x^5 + \cdots \\
&= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots
\end{aligned} \tag{4}$$

Therefore, the first order approximation of $\sin x$ about $x_e = 0$ is $\sin x \approx x$. □

2.2 Multivariable Function

Let x be an n-dimensional real vector $x \in \mathbb{R}^n$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Let $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ be a real-valued function of x. Then, the first order approximation of $f(x)$ at $x_e \in \mathbb{R}^n$

$$\begin{aligned}
f(x) &= f(x_1, x_2, \ldots, x_n) \\
&= f(x_e) + \frac{\partial}{\partial x_1} f(x)\bigg|_{x=x_e} (x_1 - x_{1e}) \\
&\quad + \frac{\partial}{\partial x_2} f(x)\bigg|_{x=x_e} (x_2 - x_{2e}) + \cdots + \frac{\partial}{\partial x_n} f(x)\bigg|_{x=x_e} (x_n - x_{ne})
\end{aligned} \tag{5}$$

Example 3. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$, a function of three variables, be $f(x_1, x_2, x_3) = x_1 x_2^2$. The linearization of this function about a point $x_e = (x_{1e}, x_{2e}, x_{3e})$ would be

$$f(x) \approx f(x_{1e}, x_{2e}) + \frac{\partial f}{\partial x_1}\bigg|_{x=x_e} (x_1 - x_{1e}) + \frac{\partial f}{\partial x_2}\bigg|_{x=x_e} (x_2 - x_{2e}) + \frac{\partial f}{\partial x_3}\bigg|_{x=x_e} (x_3 - x_{3e}) \tag{6}$$

Most of the work is in correctly evaluating the terms in blue. We can ignore writing the term for x_3 (in red in (6)) from the beginning, since $f(x)$ does not depend on x_3, meaning $\frac{\partial f}{\partial x_3} = 0$. In other words, you would be fine starting with

$$f(x) \approx f(x_{1e}, x_{2e}) + \left.\frac{\partial f}{\partial x_1}\right|_{x=x_e} (x_1 - x_{1e}) + \left.\frac{\partial f}{\partial x_2}\right|_{x=x_e} (x_2 - x_{2e}), \tag{7}$$

provided you state that this expression comes from $\frac{\partial f}{\partial x_3} = 0$.

We can derive

$$\frac{\partial f}{\partial x_1} = \frac{\partial (x_1 x_2^2)}{\partial x_1} = x_2^2 \implies \left.\frac{\partial f}{\partial x_1}\right|_{x=x_e} = x_{2e}^2 \tag{8}$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial (x_1 x_2^2)}{\partial x_2} = 2x_1 x_2 \implies \left.\frac{\partial f}{\partial x_2}\right|_{x=x_e} = 2x_{1e}x_{2e} \tag{9}$$

Therefore,

$$f(x) \approx f(x_{1e}, x_{2e}) + \left.\frac{\partial f}{\partial x_1}\right|_{x=x_e} (x_1 - x_{1e}) + \left.\frac{\partial f}{\partial x_2}\right|_{x=x_e} (x_2 - x_{2e}) \tag{10}$$

$$= x_{1e}x_{2e}^2 + x_{2e}^2(x_1 - x_{1e}) + 2x_{1e}x_{2e}(x_2 - x_{2e}) \tag{11}$$

Suppose now that you needed to rewrite this equation in terms of δx, where $\delta x = x - x_e$. As we see, it's easy to achieve this, since the only terms containing x are already in the form $x - x_e$:

$$f(x) \approx x_{1e}x_{2e}^2 + x_{2e}^2(x_1 - x_{1e}) + 2x_{1e}x_{2e}(x_2 - x_{2e}) \tag{12}$$

$$= x_{1e}x_{2e}^2 + x_{2e}^2 \delta x_1 + 2x_{1e}x_{2e}\delta x_2 \tag{13}$$

Finally, compare where we started and where we ended:

$$f(x) \approx f(x_{1e}, x_{2e}) + \left.\frac{\partial f}{\partial x_1}\right|_{x=x_e} (x_1 - x_{1e}) + \left.\frac{\partial f}{\partial x_2}\right|_{x=x_e} (x_2 - x_{2e}) \tag{14}$$

$$f(\delta x, x_e) \approx x_{1e}x_{2e}^2 + x_{2e}^2 \delta x_1 + 2x_{1e}x_{2e}\delta x_2 \tag{15}$$

□

Example 4 (Example 3 with derivative)**.** In Example 3, we had three dependent variables x_1, x_2, and x_3. Suppose that $x_1 = q$, and $x_2 = \dot{q}$. Since x_e will correspond to an equilibrium in our problems, we always know that $x_{2e} = \dot{q}_e = 0$. Of course, x_{1e} could have non-zero values. With this information, we see that the linearization becomes

$$f(x) \approx 0,$$

even though we never calculated what x_{1e} is! □

3 Linearizing ODEs

How do we linearize a nonlinear differential equation?

In general, we are interested in behavior close to equilibria, and therefore we linearize ODEs at their equilibria. Therefore,

- Find all unforced equilibria (when $u(t) \equiv 0$).
- Linearize the ODE be linearizing each term in the resulting ODE.
- For each equilibrium, rewrite the linearized equation in terms of deviations (**relative displacements**) from that equilibrium.

Example 5. Consider the input-output ODE

$$\dot{y} + y - y^3 = u.$$

Step 1: Find unforced equilibria:

$$\underbrace{\dot{y}}_{\to 0} + y_e - y_e^3 = 0 \tag{16}$$

$$\implies y_e(1 - y_e^2) = 0 \tag{17}$$

$$\implies y_e \in \{-1, 0, 1\} \tag{18}$$

Step 2: Rewrite ODE by linearizing nonlinear terms.

$$\dot{y} + y - y^3 = u \tag{19}$$

$$\implies \dot{y} + y - \left(y_e^3 + \frac{\mathrm{d}}{\mathrm{d}y}(y^3)\bigg|_{y=y_e} (y - y_e) \right) = u \tag{20}$$

$$\implies \dot{y} + y - \left(y_e^3 + 3y^2\big|_{y=y_e} (y - y_e) \right) = u \tag{21}$$

$$\implies \dot{y} + y - \left(y_e^3 + 3y_e^2(y - y_e) \right) = u \tag{22}$$

Step 3: Rewrite the equation using the perturbed variable $\delta y = y - y_e$, for each equilibria:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta y + y_e) + (\delta y + y_e) - \left(y_e^3 + 3y_e^2 \delta y \right) = u(t) \tag{23}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\delta y + \delta y - 3y_e^2 \delta y + \underbrace{\left(y_e - y_e^3 \right)}_{=0} = u(t) \tag{24}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\delta y + (1 - 3y_e^2)\delta y + 0 = u(t) \tag{25}$$

So, we get

$$
\begin{aligned}
y_e = -1: &\quad \delta\dot{y}(t) - 2\delta y = 0 && (26)\\
y_e = 0: &\quad \delta\dot{y}(t) + \delta y = 0 && (27)\\
y_e = 1: &\quad \delta\dot{y}(t) - 2\delta y = 0 && (28)
\end{aligned}
$$

□

Example 6. Consider the pendulum which is governed by the equation

$$ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = \tau(t).$$

Find all unforced equilibria, and linearize about each equilibrium.

Assume that $m = 1$ kg, $l = 1m$, and $g = 9$ m/s^2. Determine the stability of each linearized ODE. □

<u>Step 1:</u> Find equilibria

$$
\begin{aligned}
&ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = \tau(t) \\
\implies &ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = 0 && \text{(unforced)}\\
\implies &0 + mgl\sin\theta_e = 0 && \theta(t)\equiv\theta_e \implies \dot{\theta}(t)\equiv 0\\
\implies &\theta_e = \{0, \pm\pi, \pm 2\pi, \dots\}
\end{aligned}
$$

Since $\theta + 2\pi$ is the same as θ, we can assume that the set of equilibria contains just two elements: $\theta_e = \{0, \pi\}$.

<u>Step 2:</u> Linearize ODE at equilibrium θ_e

$$
\begin{aligned}
&ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = \tau(t)\\
\implies &ml^2\ddot{\theta}(t) + mgl\left(\sin\theta_e + \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)\bigg|_{\theta=\theta_e}(\theta-\theta_e)\right) = \tau(t) && \text{(first-order approx of sin)}\\
\implies &ml^2\ddot{\theta}(t) + mgl\left(\sin\theta_e + \cos\theta|_{\theta=\theta_e}(\theta-\theta_e)\right) = \tau(t) && \left(\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta) = \cos\theta\right)\\
\implies &ml^2\ddot{\theta}(t) + mgl\cos\theta_e(\theta-\theta_e) = \tau(t) && (mgl\sin\theta_e = 0)
\end{aligned}
$$

Step 3: Evaluate at equilibria, using perturbed coordinates $\delta\theta = \theta - \theta_e$:

$$ml^2 \frac{\mathrm{d}^2}{\mathrm{d}^2 t}(\delta\theta + \theta_e) + mgl\cos\theta_e(\delta\theta + \theta_e - \theta_e) = \tau(t) \tag{29}$$

$$\implies ml^2 \delta\ddot{\theta}(t) + mgl\cos\theta_e \delta\theta = \tau(t) \tag{30}$$

$$\theta_e = 0: \quad ml^2 \delta\ddot{\theta}(t) + mgl \cdot 1 \cdot \delta\theta = \tau(t) \tag{31}$$

$$\theta_e = \pi: \quad ml^2 \delta\ddot{\theta}(t) + mgl \cdot (-1) \cdot \delta\theta = \tau(t) \tag{32}$$

Stability $\theta_e = 0$: If $u(t) = \tau(t)$, $y(t) = \delta\theta(t)$. For $\theta_e = 0$,

$$(ml^2 s^2 + mgl)\hat{y}(s) = \hat{u}(s) \tag{33}$$

$$\implies G(s) = \frac{1}{ml^2 s^2 + mgl} = \frac{1}{s^2 + 3^2} \tag{34}$$

Poles are $\pm j3 \implies$ Lyapunov stable, but not asymptotically stable.

Stability $\theta_e = \pi$: If $u(t) = \tau(t)$, $y(t) = \delta\theta(t)$. For $\theta_e = \pi$,

$$(ml^2 s^2 - mgl)\hat{y}(s) = \hat{u}(s) \tag{35}$$

$$\implies G(s) = \frac{1}{ml^2 s^2 - mgl} = \frac{1}{s^2 - 3^2} \tag{36}$$

Poles are $\pm 3 \implies$ unstable.

Example 7. Consider the nonlinear EoM

$$\ddot{q}_1 + \dot{q}_2 q_2 + \cos\dot{q}_1 + q_1 = 0 \tag{37}$$

$$\ddot{q}_2 + q_1 q_2 + \dot{q}_2^2 = u \tag{38}$$

Find all equilibria $q_e = \begin{bmatrix} q_{1e} & q_{2e} \end{bmatrix}^T$.

Find the linearized ODEs at these equilibria.

Solution: Since the system is second-order in both q_1 and q_2, an equilibrium configuration q_e corresponds to equilibrium state $x_e = \begin{bmatrix} q_{1e} & q_{2e} & 0 & 0 \end{bmatrix}^T$
Step 1: Find equilibria. Set $q(t) \equiv q_e \implies \dot{q}(t) \equiv 0$ and $u(t) \equiv 0$.

$$0 + 0 \cdot q_{2e} + \cos 0 + q_{1e} = 0 \implies q_{1e} = -1 \tag{39}$$

$$0 + q_{1e} q_{2e} + 0^2 = 0 \implies q_{2e} = 0 \tag{40}$$

The unique equilibrium is $q_e = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$.

Step 2: Linearize ODEs

$$\ddot{q}_1 + \dot{q}_2 q_2 + \cos\dot{q}_1 + q_1 = 0 \tag{41}$$

$$\ddot{q}_2 + q_1 q_2 + \dot{q}_2^2 = u \tag{42}$$

$$\begin{aligned}\dot{q}_2 q_2 &= \dot{q}_2 q_2\big|_{q=q_e,\dot{q}=0} + \frac{\partial}{\partial \dot{q}_2}(\dot{q}_2 q_2)\bigg|_{q=q_e,\dot{q}=0}(\dot{q}_2 - \dot{q}_{2e}) + \frac{\partial}{\partial q_2}(\dot{q}_2 q_2)\bigg|_{q=q_e,\dot{q}=0}(q_2 - q_{2e}) \\ &= 0 \cdot q_{2e} + q_2\big|_{q=q_e}(\dot{q}_2 - 0) + \dot{q}_{2e}\big|_{q=q_e,\dot{q}=0}(q_2 - q_{2e}) \\ &= 0 + q_{2e}\dot{q}_2 + 0(q_2 - q_{2e}) \\ &= 0 + 0 \cdot \dot{q}_2 + 0 \\ &= 0\end{aligned} \tag{43}$$

$$\begin{aligned}\cos\dot{q}_1 &= \cos\dot{q}_1\big|_{q=q_e,\dot{q}=0} + \frac{\partial}{\partial \dot{q}_1}\cos\dot{q}_1\bigg|_{q=q_e,\dot{q}=0}(\dot{q}_1 - \dot{q}_{1e}) \\ &= \cos 0 + (-\sin\dot{q}_1)\big|_{q=q_e,\dot{q}=0}(\dot{q}_1 - \dot{q}_{1e}) \\ &= 1 - 0 \cdot (\dot{q}_1 - \dot{q}_{1e}) = 1\end{aligned} \tag{44}$$

$$\begin{aligned}q_1 q_2 &= q_{1e} q_{2e} + \frac{\partial}{\partial q_1}(q_1 q_2)\bigg|_{q=q_e,\dot{q}=0}(q_1 - q_{1e}) + \frac{\partial}{\partial q_2}(q_1 q_2)\bigg|_{q=q_e,\dot{q}=0}(q_2 - q_{2e}) \\ &= 0 + q_2\big|_{q=q_e}(q_1 - q_{1e}) + q_1\big|_{q=q_e,\dot{q}=0}(q_2 - q_{2e}) \\ &= 0 + q_{2e}(q_1 - q_{1e}) + q_{1e}(q_2 - q_{2e}) \\ &= 0 + 0 \cdot (q_1 - q_{1e}) + (-1) \cdot (q_2 - q_{2e}) = -(q_2 - q_{2e})\end{aligned} \tag{45}$$

$$\begin{aligned}\dot{q}_2^2 &= \dot{q}_2^2\big|_{q=q_e,\dot{q}=0} + \frac{\partial}{\partial \dot{q}_2}(\dot{q}_2^2)\bigg|_{q=q_e,\dot{q}=0}(\dot{q}_2 - \dot{q}_{2e}) \\ &= 0 + 2\dot{q}_2\big|_{q=q_e,\dot{q}=0}(\dot{q}_2 - 0) \\ &= 0\end{aligned} \tag{46}$$

Putting it all together, we get

$$\ddot{q}_1 + 0 + 1 + q_1 = 0 \tag{47}$$
$$\ddot{q}_2 + -(q_2 - q_{2e}) + 0 = u \tag{48}$$

Step 3: Let $\delta q_1 = q_1 - q_{1e} = q_1 - 1$ and $\delta q_2 = q_2 - q_{2e} = q_2$. Substituting for q in the linearized ODE, we get

$$\delta\ddot{q}_1 + 0 + 1 + (\delta q_1 - 1) = 0 \tag{49}$$
$$\delta\ddot{q}_2 + -(\delta q_2 + q_{2e} - q_{2e}) + 0 = u, \tag{50}$$

or

$$\delta\ddot{q}_1 + \delta q_1 = 0 \tag{51}$$
$$\delta\ddot{q}_2 - \delta q_2 = u, \tag{52}$$

□

Example 8. Consider the EoM

$$\ddot{q}_1 + 4\dot{q}_1 q_2 - q_2 + 1 = 0 \tag{53}$$
$$\ddot{q}_2 + 10q_1 - \frac{3}{\pi}\sin(\pi q_2) = 34u \tag{54}$$

1. Find the unforced equilibria (q_{1e}, q_{2e})
2. Find the linearized ODEs at these unforced equilibria
3. Let $y = q_1 - q_{1e}$. Find $G(s) = \hat{y}(s)/\hat{u}(s)$
4. Find the impulse response of $G(s)$
5. Find the steady-state forced response $y_{ss}(t)$ to input $u(t) = \cos t$.

Solution:

1. $(q_{1e}, q_{2e}) = (0, 1)$
2. $$\ddot{\delta}q_1 + 4\delta\dot{q}_1 - \delta q_2 = 0 \tag{55}$$
$$\ddot{\delta}q_2 + 10\delta q_1 - 3\delta q_2 = 34u \tag{56}$$
3. $$G(s) = \frac{34}{s^3 + 7s^2 + 12s + 10}$$
4. $$y_i(t) = 2e^{-5t} - 2e^{-t}\cos t + 8e^{-t}\sin t$$
5. $$y_{ss}(t) = \frac{34}{\sqrt{130}}\cos\left(t - \tan^{-1}(187/51)\right)$$

□