ME 340 Fall 2020
Solving Differential Equations

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Solutions to ODEs

Goal

Learn how to solve a linear ODE

$$a_n y^n(t) + a_{n-1} y^{n-1}(t) + \cdots a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = f(t),$$

given appropriate initial conditions $y(0)$, $\dot{y}(0)$ $\dots$, $y^{n-1}(0)$.

Solution

A function $y(t)$ that will satisfy the constraints given by the linear ODE and the initial conditions.

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Approaches

1. Guess time-based functions
2. Invert frequency-based functions

Time-based Approaches

Solutions have two components

The solution $y(t)$ has two parts:

$$y(t) = y_H(t) + y_P(t),$$

where $y_H(t)$ is the homogenous solution, and $y_P(t)$ is the particular solution.

Time-based Approaches

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Basic idea

- $y_H(t)$ is the solution of

$$a_n y^n(t) + a_{n-1} y^{n-1}(t) + \cdots a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0,$$

  the unforced equation.
- use $y_H(t)$ to solve for $y_P(t)$, through the method of undetermined coefficients.

Homogenous Solutions

Assume $y(t) = ke^{\lambda t}$. From calculus,

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}.$$

Applying this to

$$a_n y^n(t) + a_{n-1} y^{n-1}(t) + \cdots a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0,$$

we get

$$a_n \lambda^n y(t) + a_{n-1} \lambda^{n-1} y(t) + \cdots a_2 \lambda^2 y(t) + a_1 \lambda y(t) + a_0 y(t) = 0,$$

or,

$$\left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots a_2 \lambda^2 + a_1 \lambda + a_0\right) y(t) = 0.$$

Homogenous Solutions

Since $y(t) = ke^{\lambda t}$, $y(t) \neq 0$ for any t.

To solve $y(t)$, we must solve for λ in

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$$\boxed{a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_2\lambda^2 + a_1\lambda + a_0 = 0.}$$

Characteristic equation

The characteristic equation is a polynomial (algebraic) equation with real coefficients.

Its solutions are possibly complex numbers, that is, $\lambda_i \in \mathbb{C}$, for $i \in \{1, 2, \ldots n\}$.

Each such complex number is a called a root of the characteristic equation, and each root adds a term to the solution $y_H(t)$ (next slide).

Roots of Characteristic Equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

Characteristic equation

Distinct roots $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$.

$$y_H(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + \cdots + k_n e^{\lambda_n t}$$

Repeated roots $\lambda_1 = \lambda_2 = \lambda_3, \lambda_4 \neq \cdots \neq \lambda_n$.

$$\begin{aligned} y_H(t) &= k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_2 t} + k_3 t^2 e^{\lambda_2 t} + k_4 e^{\lambda_4 t} + \cdots + k_n e^{\lambda_n t} \\ &= (k_1 + k_2 t + k_3 t^2) e^{\lambda_1 t} + k_4 e^{\lambda_4 t} + \cdots + k_n e^{\lambda_n t} \end{aligned}$$

Complex roots $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta, \lambda_3, \lambda_4, \ldots, \lambda_n$.

$$\begin{aligned} y_H(t) &= k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + \cdots + k_n e^{\lambda_n t} \\ &= k_1 e^{(\alpha+i\beta)t} + k_2 e^{(\alpha-i\beta)t} + \cdots + k_n e^{\lambda_n t} \\ &= e^{\alpha t}(k_a \cos(\beta t) + k_b \sin(\beta t)) + \cdots + k_n e^{\lambda_n t} \end{aligned}$$

Particular Solution

We found the form of $y_H(t)$, which is one part of the solution $y(t)$ to the ODE

$$a_n y^n(t) + a_{n-1} y^{n-1}(t) + \cdots + a_0 y(t) = f(t).$$

We use the form of the forcing function $f(t)$, and of $y_H(t)$, to predict a candidate particular solution $y_P(t)$.

$$\text{form of } f(t) \text{ and } y_H(t) \implies \text{form of } y_P(t)$$

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Candidate Particular Solution

Forcing $f(t)$	Candidate $y_P(t)$
α	A
$\alpha t + \beta$	$At + B$
$e^{\alpha t}$	$Ae^{\alpha t}$
$\cos \omega t$	$A \cos \omega t + B \sin \omega t$
$\sin \omega t$	$A \cos \omega t + B \sin \omega t$

Another rule

If candidate $y_P(t)$ contains a term that is present in $y_H(t)$, multiple that term by t.

Example

if $y_H(t) = k_1 e^{-t} + k_2 e^{-2t}$, and $f(t) = e^{-t}$, use candidate

$$y_P(t) = A\ t\ e^{-t}.$$

If candidate $y_H(t) = k_1 e^{-t} + k_2 t e^{-t}$, due to repeated roots, use $y_P(t) = A\ t^2\ e^{-t}$.

Are we there yet?

- Find homogenous solution $y_H(t)$, where we ignore $f(t)$.
- Use $f(t)$ and $y_H(t)$ to choose candidate $y_P(t)$.
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Final Step

1. Apply initial conditions to our candidate solution $y(t) = y_H(t) + y_P(t)$, obtain equations in coefficients k_i
2. Solve for k_i
3. If we get valid solutions, we're done
4. Otherwise, our candidates were bad.

Example

Solve

$$\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = \cos 2t, \quad y(0) = 0, \quad \dot{y}(0) = 0.$$