# Probability

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## 1 Sigma Algebra

**Definition 1** (Sigma Algebra)**.** A $\sigma$-algebra (aka $\sigma$-field) on a set $X$ is a collection $\Sigma$ of subsets of $X$ that

1. $X \in \Sigma$: includes $X$ itself,
2. $A \in \Sigma \implies A^c \in \Sigma$: is **closed under complement**, and
3. $A_1, A_2 \in \Sigma \implies A_1 \cup A_2 \in \Sigma$: is **closed under countable unions**.

**Example 1.** The power set of a countable set $X$ forms a $\sigma$-algebra over $X$.

**Example 2.** If $X = \{a, b, c, d\}$, one possible $\sigma$-algebra on $X$ is $\Sigma = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$

**Example 3.** If $\{A_1, A_2, \dots\}$ is a countable partition of $X$ then the collection of all unions of sets in the partition (including the empty set) is a $\sigma$-algebra.

**Example 4** (Borel $\sigma$-algebra of $X$)**.** Set of all results of countable union, countable intersection, and relative complement applied to the open sets of a topological space $X$.

## 2 Measure & Probability Spaces

**Definition 2** (Measurable Space)**.** A measurable space is a set $X$ to which we can assign a $\sigma$-algebra.

**Example 5** (Real numbers)**.** The real numbers $\mathbb{R}$ with the Borel $\sigma$-algebra forms a measurable space.

The space $(X, \Sigma)$ is measurable because the properties of the $\sigma$-algebra $\Sigma$ will enable us to map elements of $\Sigma$ to $\mathbb{R}_{\geq 0} \cup \{\infty\}$ so that disjoint union in $\Sigma$ is equivalent to addition in $\mathbb{R}$.

**Definition 3.** Measure A measure $\mu$ is a mapping from a $\sigma$-algebra to $\mathbb{R}$ if

- For all $E \in \Sigma$, $\mu(E) \geq 0$
- $\mu(\emptyset) = 0$
- for pair-wise disjoint sets $E_k \in \Sigma$, $\mu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$

**Example 6** (Discrete Measure)**.**

**Example 7** (Borel Measure)**.**

**Definition 4.** Probability Measure A measure $\mu$ is a probability measure on $X$ if

$$\mu(X) = 1.$$

**Definition 5** (Measure Space)**.** A measure space $(X, \Sigma, \mu)$ is a measurable space $(X, \Sigma)$ equipped with a measure $\mu\colon X \to \mathbb{R}$.

**Definition 6** (Probability Space)**.** A probability space $(\Omega, \mathcal{F}, P)$ is a set of outcomes $\Omega$ on which one defines the set of outcomes $\mathcal{F}$ which is a $\sigma$-algebra, and equips the resulting measurable space $(\Omega, \mathcal{F})$ with a probability measure $P$.

# 3 Random Variable

**Definition 7** (Measurable Function)**.** A function $f \in \Sigma_1 \to \Sigma_2$ is measurable if $\Sigma_1$ and $\Sigma_2$ are $\sigma$-algebras defined over appropriate spaces, and the preimage of elements in $\Sigma_2$ are members of $\Sigma_1$.

**Definition 8** (Random Variable)**.** A random variable $X$ is a measurable function from a probability space $(\Omega, \mathcal{F}, P)$ to a measurable space $(E, \mathcal{E})$.

So, why do we need the second measurable space if we already have a probability (measure) space?

We don't.

We need the first probability space when our measurable space of interest doesn't come with an explicit probability measure.

*Remark* 1. Almost all elementary probability problems are asking you to construct the pre-image of an event so that you can apply known probabilities.

**Example 8.** When asked for the probability of the the sum of face-up numbers on two die equaling six. The set of outcomes we are interested in are the set of non-negative integers $\mathbb{N}$. Since this set is countable $\mathbb{N}$, we can define a $\sigma$-algebra for it and get a measurable space.
How do we assign the probability measure over this set? We use the probability measure for rolls of two die, by defining a measurable mapping from that set of outcomes to the set $\mathbb{N}$.

*Remark* 2. This mapping from the rolls of two die to $\mathbb{N}$ is called the random variable corresponding to the sum of the face-up numbers on two die. While not obvious, what this means is the set of outcomes under that mapping is the random variable. For this set of outcomes to possess a meaningful probability, it must belong to some measurable space.

**Definition 9** (Random Variable In Words)**.** A random variable is a measurable space of events equipped with a probability measure given directly or through a different probability space combined with a measurable mapping to the events of interest.

# References

[1] Author; Title; Publisher.