# 1 Stability Of A System

Consider the simple pendulum at its downward position, with zero velocity. The ouput is the angle $\theta$, which is zero as shown

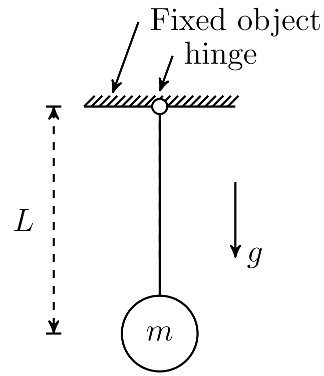


Figure 1: Pendulum in downward equilibrium position.

A quick sidewards tap on the mass is known as providing an *impulse*. We're sure that the pendulum moves away from the downward position in response to the tap. But what will happen in the long run? Here, we're asking for the **impulse response** of the simple pendulum.

For any input or initial conditions, there are three possible behaviors of the impulse response $y_i(t) = \mathcal{L}^{-1}\{G(s)\}$:

1. $y_i(t)$ is unbounded ($|y_i(t)| \to \infty$)
2. $y_i(t)$ is bounded (We can find $0 < M < \infty$ such that $|y_i(t)| \leq M$ for all $t$)
3. $\lim_{t\to\infty} y_i(t) = 0$

We can use these three behaviors to define three notions of stability:

**Definition 1** (Unstable)**.** $G(s)$ is unstable (US) if its impulse response is unbounded.

**Definition 2** (Lyapunov Stable)**.** $G(s)$ is Lyapunov stable (LS) if its impulse response is bounded.

**Definition 3** (Asymptotically Stable)**.** $G(s)$ is asymptotically stable (AS) if its impulse response satisfies $\lim_{t\to\infty} y_i(t) = 0$.

**Note:** An asymptotically stable TF is Lyapunov stable. An unstable system is not LS, and therefore not AS either.

**Remark:** Why are we interested in $y_i(t) \to 0$ instead of $y_i(t) \to a$, where $a \neq 0$? The answer is that we assume we are interested in equilibria, and for a linear system, 0 is its equilibrium.

## 1.1 Stability and Poles of the Transfer Function

Let's apply the notion of multiplicity of roots, first mentioned in Laplace transforms, to the multiplicity of poles.

**Definition 4** (Multiplicity)**.** Let $G(s) = \beta(s)/\alpha(s)$.
If $\alpha(p) = 0$,

$$\lim_{s \to p} \frac{\alpha(s)}{(s-p)^n} \neq 0, \text{ and } \lim_{s \to p} \frac{\alpha(s)}{(s-p)^{n-1}} = 0,$$

then $p$ is a pole of $G(s)$ with multiplicity $n$.

**Example 1.** Let

$$G(s) = \frac{1}{(s-2)(s-1)^2 s^3}.$$

Now, $\alpha(s) = (s-2)(s-1)^2 s^3$. By our definition above,

- $p_1 = 2$ is a pole of $G(s)$ with multiplicity 1
- $p_2 = 1$ is a pole with multiplicity 2
- $p_3 = 0$ is a pole with multiplicity 3

**Fact:** $G(s)$ is asymptotically stable (AS) if and only if all its poles are in the open left half plane (OLHP).

**Fact:** $G(s)$ is Lyapunov stable (LS) if all its poles either

- are in the OLHP, or
- are on the imaginary axis (IA) with multiplicity 1.

**Fact:** G(s) is unstable (US) if has a pole

- in the open right half plane (OHRP), or
- on the IA with multiplicity greater than 1.

These facts above are one example of the idea studying $G(s)$ tells us something about the responses of a system to a given set of inputs.

**Example 2.** Consider a system with transfer function $G(s)$ given by

$$G(s) = \frac{s}{s^2 + 5s + 6}. \tag{1}$$

Classify the stability properties of this sytem.

**Solution:** The denominator polynomial is $s^2 + 5s + 6$. The poles of $G(s)$ are therefore the solutions (or, roots) of the equation $s^2 + 5s + 6 = 0$, which turn out to be $p_1 = -2, p_2 = -3$ (we can also say $p_1 = -3, p_2 = -2$). The real part of both these roots are in the OLHP, therefore $G(s)$ is asymptotically stable. Since $G(s)$ is asymptotically stable (AS), it is also Lyapunov stable (LS).

**Example 3.** Consider a system with transfer function $G(s)$ given by

$$G(s) = \frac{s}{s^2 - 6s + 5}. \tag{2}$$

Classify the stability properties of this sytem.

**Solution:** The denominator polynomial is $s^2 - 6s + 5$. The poles of $G(s)$ are therefore the solutions (or, roots) of the equation $s^2 - 6s + 5 = 0$, which turn out to be the complex conjugate pair $p_{1,2} = 3 \pm j2$. The real part of both these roots are strictly positive, the poles are in the ORHP, therefore $G(s)$ is unstable (US).

## 1.2 Initial and Final Value Theorems

In some case, we may only want to know the value of $y(t)$ at specific times of interest, and solving for $y(t)$ using the inverse Laplace transform is involved. For example, consider a system that is stable, but not asymptotically stable. Then, $y(t)$ remains bounded, and if it approaches a constant value, we'd like to know what that value is. We may be able to calculate this value without ever solving for $y(t)$. The Final Value Theorem helps us do this.

### 1.2.1 Final Value Theorem

Let $y(t)$ have the Laplace transform $\hat{y}(s)$ (which could be a response of the form $G(s)\hat{u}(s)$). If the poles of $\hat{y}(s)$ are in the OLHP with the possible exception of a single pole at zero. Then,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s). \tag{3}$$

### 1.2.2 Initial Value Theorem

A similar result let's us know what the initial value is.

$$y(0) = \lim_{t\to 0} y(t) = \lim_{s\to\infty} s\hat{y}(s). \tag{4}$$

**Example 4.** Consider the response of a second order system to a step input:

$$\hat{y}_{step}(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)},$$

where $\xi > 0$ Find the initial and final value of the response.
**Solution:**

$$\begin{aligned} y_{step}(0) &= \lim_{s\to\infty} s\hat{y}_{step}(s) && (5)\\ &= \lim_{s\to\infty} s\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} && (6)\\ &= \lim_{s\to\infty} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} && (7)\\ &= 0 && (8) \end{aligned}$$

Since $\xi > 0$, two poles are in the open left half plane, and one on the imaginary axis. Therefore, we may use the FVT to calcluate $y_{step}(\infty)$.

$$\begin{aligned} y_{step}(\infty) &= \lim_{s\to 0} s\hat{y}(s) && (9)\\ &= \lim_{s\to 0} s\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} && (10)\\ &= \lim_{s\to 0} \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} && (11)\\ &= 1 && (12) \end{aligned}$$

If we have a mass-spring-damper at equilibrium, and apply a step input force $f(t)$ on the mass, the response of the position of the mass $y = q$ is

$$\hat{y}(s) = \frac{1}{(ms^2 + cs + k)}\frac{1}{s} \implies y_{step}(\infty) = \frac{1}{k} \tag{13}$$

We've just shown that a stiffer spring (higher $k$) reduces the distance (smaller $1/k$) by which a constant force (step input $f(t)$) moves the resting position ($0 \to \frac{1}{k}$).