Differential Geoemtry

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Contents

1 Smooth Manifold

A manifold Q is a second-countable Hausdorff space with a set of compatible charts that cover Q. The charts are local diffeomorphisms to $\mathbb{R}^n$, the compatibility makes the charts an atlas. One either uses chart ϕ to map an open set of Q to its coordinates in $\mathbb{R}^n$, or to assign an open set of points on the manifold to an open set of $\mathbb{R}^n$, which effectively defines coordinates.

2 CoTangent And Tangent Space

The tangent bundle TQ is the collection of tangent spaces T_qQ at each $q \in Q$.The tangent space is an affine space.

A natural basis to assign to T_qQ is one derived from the chart ϕ. Essentially, a natural basis vector $e^i \in T_qQ$ corresponds to the tangent to the straight line through the point $\phi(q)$ aligned with the i^{th} coordinate axis. The natural basis for the tangent space induces a metric on T_qQ that is consistent locally with the Euclidean metric on $\mathbb{R}^n$, the coordinates near q.

Since T_qQ is a vector space, we may define the space of linear functionals on T_qQ as the cotangent space T_q^*Q (another n-dimensional vector space). A natural basis for T_q^*Q is the set of n linear functions whose evaluations of the n-basis vectors of T_qQ form the Kronecker delta function. In effect, this natural definition is an analogue to the *reciprocal basis* of a vector space basis. Under the usual outer product, this analogy becomes equivalence.

Example 1. Consider the basis, colored blue, in Figure 1. To define a reciprocal basis that meets the requirements of a Kronecker delta function upon taking inner products, we need to rotate e_A^1 clockwise, and e_A^2 anti-clockwise. In general we may also need to scale the resulting rotated vectors, however in this case we do not.

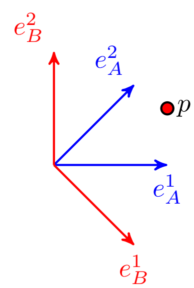

Figure 1: The reciprocal basis B (red) to basis A (blue). If A was orthonormal, then B would be the same as A. These oblique coordinates demonstrate the distinction from the usual case of orthonormal bases.

The significance of the reciprocal basis in the tangent space that also defines a basis for the dual cotangent space is as follows:

1. An element of the reciprocal basis defines a direction of motion that does not change Euclidean projection on planes corresponding to other coordinates (other covariant coordinates).
2. The exterior product of a vector v with an element of the cotangent basis ω^i is invariant to addition to v from the span of the basis of the tangent space with e^i removed.

2.1 Basis Transformations

Let's say we equip $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ with bases B_1 and B_2 respectively. Let the metrics on $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$, and the outer product, be the usual ones. So, if we have representations v, for $v \in \mathbb{R}^n$ and ω for $\omega \in (\mathbb{R}^n)^*$, we get

$$\|v\|_I^2 = v^T v \tag{1}$$
$$\|\omega\|_I^2 = \omega^T \omega \tag{2}$$
$$\langle \omega, v \rangle = \omega^T v \tag{3}$$

Suppose we transform $\mathbb{R}^n$ by a linear transformation represented as T_1 in B_1 and $(\mathbb{R}^n)^*$ by a linear transformation represented as T_2 in B_2. Then, we would see that the $v \mapsto T_1^{-1} v$ and $\omega \mapsto T_2^{-1}\omega$.

$$\|v'\|_I^2 = v^T T_1^{-T} T_1^{-1} v \tag{4}$$
$$\|\omega'\|_I^2 = \omega^T T_2^{-T} T_2^{-1} \omega \tag{5}$$
$$\langle \omega', v' \rangle = \omega^T T_2^{-T} T_1^{-1} v \tag{6}$$

So, if we want the outer product to be consistent after the change of bases, we need $T_2 = T_1^{-1}$. Let $M^{-1} = T_1^T T_1$. Then,

$$\|v'\|_I^2 = v^T T_1^{-T} T_1^{-1} v = \|v\|_M^2 \tag{7}$$
$$\|\omega'\|_I^2 = \omega^T T_1^T T_1 \omega = \|\omega\|_{M^{-1}}^2 \tag{8}$$
$$\langle \omega', v' \rangle = \omega^T v \tag{9}$$

2.2 Contra and Covariance

If we change the basis of a tangent space through a map T, how should we change the components of a representation v so that we are referring to the same vector in both representation? Recall that the columns of the representation of T as a matrix are the new basis expressed in terms of the old basis.

Coordinates require us to orthogonally project points to subspaces. The main question is how we define as orthogonal. If we use the basis itself to define orthogonality, we obtain contravariant coordinates, and orthogonal projection lines are also parallel to basis vectors. Essentially, projection respects the basis vectors and is perpendicular in their own basis. If we use orthogonality in the Cartesian sense, then we obtain covariant coordinates $[v]_{cov}$.

There are two interpretations:

- Orthogonal (Cartesian) projection onto the non-orthogonal basis vectors
- Orthogonal (basis-defined) projection in the *conjugate* basis

Contravariant coordinates allow reconstruction by vector space operations, but not covariant coordinates.

If you transform the basis in some way, covariant coordinates change the same way, contravariant the opposite. In effect, if we transform the basis by map with representation T, then a covariant coordinate $[v]_{cov}$ in the old basis becomes the $T[v]_{cov}$in the new basis. Similarly, $[v]_{contra}$ becomes $T^{-1}[v]_{contra}$.

3 Connections

Two nearby points on the manifold are similar, but their tangent spaces have no connection to one another. Since they are vector spaces, one can imagine infinite ways to relate them, or to connect them.

At q and $q' = q + dq$, we have tangent spaces $T_q Q$ and $T_{q'} Q$

Suppose we have $e_i \in T_{q'} Q$. We can translate this basis vector in the ambient space, and move its point of attachment to q which is the origin of $T_q Q$, although most likely this vector does not lie in $T_q Q$. So, we project the translated e_i to $e_i' \in T_q Q$.

In the end, $e_i' = e_i + de_i$, where $de_i \in T_q Q$ and has coordinates

$$de_i = (de_i^j) e_j \tag{10}$$

To a first order approximation, we want

$$de_i^j = \Gamma_{ki}^j(q) dq^k \tag{11}$$

That is, the correction between a vector in two different tangent spaces is linear in the difference between the coordinates at those two points. This linearity implies that as $dq \to 0$, $e_i' \to e_i$.

We understand affine connections, but how do we derive one?

4 Covariant Derivative

We have a vector field $X(q) = X_i(q)e^i(q)$. How do we define the derivative? It is clear that $X(q+dq) - X(q)$ makes no sense.

We have to first pull $X(q+dq)$ back to T_qQ. We do this pulling back through the coordinates.

$$\tilde{e}_i = e_i(x) + \Gamma^k_{ij}(x)e_k dx^j \tag{12}$$

$$\begin{aligned}\tilde{X} &= X^k(q+dq)\tilde{e}_k \\ &= (X^k(q) + \partial_j X^k dq_j)\tilde{e}_k \\ &= X^k(q)e_k + \partial_j X^k dq^j e_k + \Gamma^m_{jk}(q)e_m X^k dq^j\end{aligned} \tag{13}$$

where it appears that the product $\partial_j X^k dq^j$ is ignored. Now, $\tilde{X} - X$ at any q is $\partial_j X^k dq^j e_k + \Gamma^m_{jk}(q)e_m X^k dq^j$, so that the derivative has components

$$\nabla^k_i X^k = \partial_i X^k + \Gamma^k_{ij}X^j \tag{14}$$

5 Riemannian Metrics

Once we define a metric $\mathbb{G}_q\colon T_qQ \times T_qQ \mapsto \mathbb{R}$, we get an affine connection, covariant derivative, and hence geodesics. The coordinate-independent equations of a mechanical system are then

$$\nabla_{\dot{q}}\dot{q} = \mathbb{G}^\sharp_q(-dV + f + f_d)$$

where V is a conservative potential, f_d is the dissipation, and f are the external generalized forces.

Our quest is that for *any* (suitable) coordinate system, we can achieve motions once we estimate $\mathbb{G}_q$, f_d, and dV in that coordinate system. This approach opens the door to sensorimotor control.

5.1 In Coordinates

Once we define coordinates, we get the Euclidean metric for free on T_qQ through the natural basis. However, we may prefer a different size measure for vectors defined using the natural basis. This metric is the Riemannian metric $M(q)$. As the derivation in Section 2.1 shows, to choose metric $M(q)$ on T_qQ requires us to choose metric $M^{-1}(q)$ on T^*_qQ to keep the outer product identical to the Euclidean case.

The metric also leads to the sharp map $\mathbb{G}^\sharp_q\colon T^*_qQ \to T_qQ$ which maps torques (cotangents) to velocities (tangents). In effect, $F \mapsto M^{-1}(q)F \in T_qQ$. Similarly, we get the flat map $\mathbb{G}^\flat_q\colon T_qQ \to T^*_qQ$ which maps velocities to torques. In effect, $v \mapsto M(q)v \in T^*_qQ$. The point of these maps is to ensure that taking the outer product between cotangent and tangent is the same as evaluating the metric after transformation:

$$F^Tv = \langle F, v\rangle = \langle G^\sharp(F), v\rangle_M = \langle M^{-1}(q)F, v\rangle_M = F^TM^{-T}(q)M(q)v = F^Tv$$

Similarly $\langle F, v\rangle = \langle F, G^\flat(v)\rangle_{M^{-1}}$.

Alternatively, we may say that the metric on T_qQ induces a metric for T^*_qQ under the natural basis, or a basis for T^*_qQ under the natural metric, and the natural exterior product makes sense either way.

6 Errors On Manifolds

Consider two points $q \in Q$ and $r \in Q$, with the understanding of the current and reference configurations. We may define an smooth function $\phi\colon Q \times Q \to \mathbb{R}$. It is an error function if it is positive definite. In other words $\phi(q, r) \geq 0$ with equality if and only if $q = r$. It is symmetric if $\phi(q, r) = \phi(r, q)$ for all $q, r \in Q$.

6.1 Transport Map and Velocity Error

Let $\mathrm{d}_1\phi$ and $\mathrm{d}_2\phi$ denote the differentials of ϕ with respect to the first and second arguments. A map $\mathcal{T}_{(q,r}: T_rQ \to T_qQ$ is a **transport map** if it is compatible with the configuration error, meaning that

$$\mathrm{d}_2\phi = -T^*_{(q,r)}\mathrm{d}_1\phi, \tag{15}$$

where $\mathcal{T}^*_{(q,r}: T^*_qQ \to T^*_rQ$ is the dual map of $\mathcal{T}_{(q,r}$. Intuitively, this map correctly relates the steepest direction of decreasing errors at the two points q and r. For Euclidean-distance based errors, $T^*_{(q,r)} = I$ for all q, r.

Given $\dot{q} \in Q$ and a velocity $\dot{r} \in T_rQ$, the velocity error is

$$\dot{e} = \dot{q} - T_{(q,r)}\dot{r}, \tag{16}$$

where $\dot{r}$ has been transported into T_qQ.

It is then possible to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(q(t), r(t)) = \mathrm{d}_1\phi(q(t), r(t)) \cdot \dot{e}(t) \tag{17}$$

References

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