

ME/AER 676 Robot Modeling & Control

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Velocities of Frames

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- ▶ However, the coordinate (d, R) is not a vector!

Velocities in R^n

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- ▶ The **subtraction** and **division** operations make sense in a vector space
- ▶ However, if x belonged to a group, we can't define a derivative this way

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- ▶ The 'velocity' would require us to take the limit as $h \rightarrow 0$ of the ratio of $\Delta R(h)$ and some measure of the size of $\Delta R(h)$.

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- ▶ Physically, the vector $\omega = [\omega_1 \ \omega_2 \ \omega_3]^T$ defines the instantaneous angular velocity in frame $\{0\}$

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- ▶ There's a one-to-one relationship between a vector \mathbb{R}^3 and the set of 3×3 skew-symmetric matrices
- ▶ Therefore, we can represent the rate of change of orientation using an angular velocity.
- ▶ So, when a task is $x(t) = (d(t), R(t)) \in \mathbb{R}^3 \times SO(3)$, its velocity is

$$\xi \in \mathbb{R}^6 = \underbrace{\mathbb{R}^3}_{\text{linear velocity}} \times \underbrace{\mathbb{R}^3}_{\text{angular velocity}}$$

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 - ▶ As the three derivatives of the three numbers used to parametrize $SO(3)$.

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- ▶ When orientation is not three numbers, $J(q)$ is geometric
- ▶ Columns of $J(q)$ of geometric Jacobian are derived geometrically