ME/AER 676 Robot Modeling & Control
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Inverse Kinematics

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Introduction

- The Forward Kinematics problem combines known closed-form expressions for individual homogenous transformations
- No closed-form expression for f in $x = f(q)$ needs to be maintained to obtain x
- Computing the inverse, however, is not as easy
- The inverse kinematics problem is often not even unique, which has algorithmic implications

Inverse Kinematics

Since we know how to build $f(q)$, we arrive at two approaches to inverse kinematics

- Analytic approaches:
  Build the closed-form expression $f(q)$ and define a closed-form inverse $f^{-1}(q)$
- Numerical approaches:
  Numerically search for values of q so that $f(q) = x$, where the function f is either closed-form or numerical

Analytic Inverse Kinematics

- Complicated to derive, but enables fast computations
- Some robots are designed with geometries that simplify the expressions:
  - The wrist is has three links with intersecting axes of rotation (spherical joint)
  - The end-effector frame coincides with wrist center.

Numerical Inverse Kinematics

- solve optimization:

$$\min_{q} \quad \|x - f(q)\|_2^2$$

- We can add constraints that make the solution unique, or other benefits
- We may also use other measures for the distance between x and $f(q)$

Analytical Inverse Velocity Kinematics

- Instead of $q = f^{-1}(x)$, some tasks require calculating $\dot{q}$ given task space velocity ξ
- If $J(q)$ is square and full-rank, then $\dot{q} = J(q)^{-1}\xi$
- If $J(q) \in \mathbb{R}^{m \times n}$, $m < n$, and $\mathrm{rank}(J(q)) = m$, we may compute

  $$\dot{q} = J^{+}\xi + (I - J^{+}J)b,$$

  where pseudo-inverse J^{+} is

  $$J^{+} = J^{T}(JJ^{T})^{-1},$$

  and $b \in \mathbb{R}^{n}$ is an arbitrary vector that does not affect ξ.

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Numerical Inverse Velocity Kinematics

- Instead of 'closed-form' pseudo-inverse J^+, solve optimization:

$$\min_{q} \quad \|\xi - J(q)\dot{q}\|_2^2$$

- Here too, we can add constraints that make the solution unique, or other benefits
- Again, we may also use other measures for the distance between ξ and $\dot{q}$

Differential Inverse Kinematics

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$$\frac{d}{dt} L(q) = (x - f(q))^T (\xi - J(q)\dot{q}) \tag{1}$$

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$$\xi - J(q)\dot{q} = -(x - f(q)) \tag{2}$$

$$\implies \dot{q} = J^+ (\xi + (x - f(q))), \text{ and} \tag{3}$$

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- Also works as a task-space position controller, assuming a low-level velocity-tracking loop!

Euler Integration Version

- The Differential Inverse Kinematics Approach is asking us to solve the ODE

$$\dot{q}(t) = J^{+}\left(\xi(t) + (x(t) - f(q(t)))\right)$$

for $q(t)$ given $x(t)$. If $x(t) \equiv x$ (fixed), set $\xi(t) \equiv 0$.

- as $t \to \infty$, assuming J remains full rank, we expect $q(t) \to f^{-1}(x(t))$, solving IK
- Instead of ODE, we can take small steps

$$q_{k+1} = q_k + \eta J^{+}(q_k)\left(\xi_k + (x_k - f(q_k))\right)$$

for some step size η

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  - Note that we are building $q(s)$ as we go!
- The integration drifts, so we need a correction term

$$\dot{q}(t) = J^+\xi(t) + \underbrace{J^+ (x(t) - f(q(t)))}_{\text{error correction}}$$